OK Geometry Plus

Reference for OK Geometry Plus (v.17.2)

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Contents

1. Introduction

This document describes specific features of OK Geometry Plus. OK Geometry Plus is intended for advanced users who are interested in triangle geometry. Several functions that are added to the Basic module allow the realisation of constructions based on advanced geometric properties of triangles and triangle related operations. Furthermore, these properties and operations are used in the (observational) analysis of geometric configurations.

The core of OK Geometry Plus is a large database of

- triangle centres (e.g. symmedian point, currently more than 30 000)
- triangle related triangles (e.g. orthic triangle, currently more than 200)
- triangle related lines (e.g. Euler line, currently 11 lines, more that 2000 including those of related triangles)
- triangle related circles (e.g. 9-point circle, currently 36 circles, more than 5000 including those of related triangles)
- triangle related conics (e.g. Feuerbach hyperbola of a triangle, currently 18 conics)
- triangle operations that associate a point, line, circle, triangle, or conic to a given point (e.g. isogonal conjugation of a point, currently 49 operations)

All these operations can be used directly in constructions as well as in observing how an object is related to a given triangle. Besides some examples at the end we present three main way of working in Plus mode:

- Simple observation of triangle objects
- Using special triangle related commands in constructions
- Advanced observation analysis of a triangle

Constructing as well as observing in triangle geometry involves a multitude of concepts and, consequently, a large number of operations and various objects. In order to facilitate the work in triangle geometry, a help system was developed. It is explained in the next section.

2. Triangle help system

The triangle help system comprises two tools: Triangle index and Triangle glossary. Both tools are accessible in all triangle related texts in OK Geometry, including in texts inside the tools themselves.

1.1. Triangle glossary

The triangle glossary is accessible with the *Help|Triangle glossary* command or with F1 key. The glossary gives a lists of triangle concepts related to a given term. We explain the use of glossary with an example.

Suppose that we encounter the term ‘Gergonne point’. Maybe we do not know what is it about nor where to look for the related commands. We call the glossary (pressing F1 key) and write the (approximate) term we alre looking for (for example ‘gergone’). In the form a list of about 20 items appears, items are listed in decreasing relevance:

- first, the items that contain the sought term (or similar) in the title of the item,
- then, the items with – in front that contain the sought term (or similar) in the explanatory text,

- finally, with - - in front, the centres form ETC (Encyclopedia of Triangle Centres) that contain the term (or similar) in the name of the centre.

A click on any item provides more information. In our case we click the first item ('Gergonne point'), and the for is filled, see Figure 1.

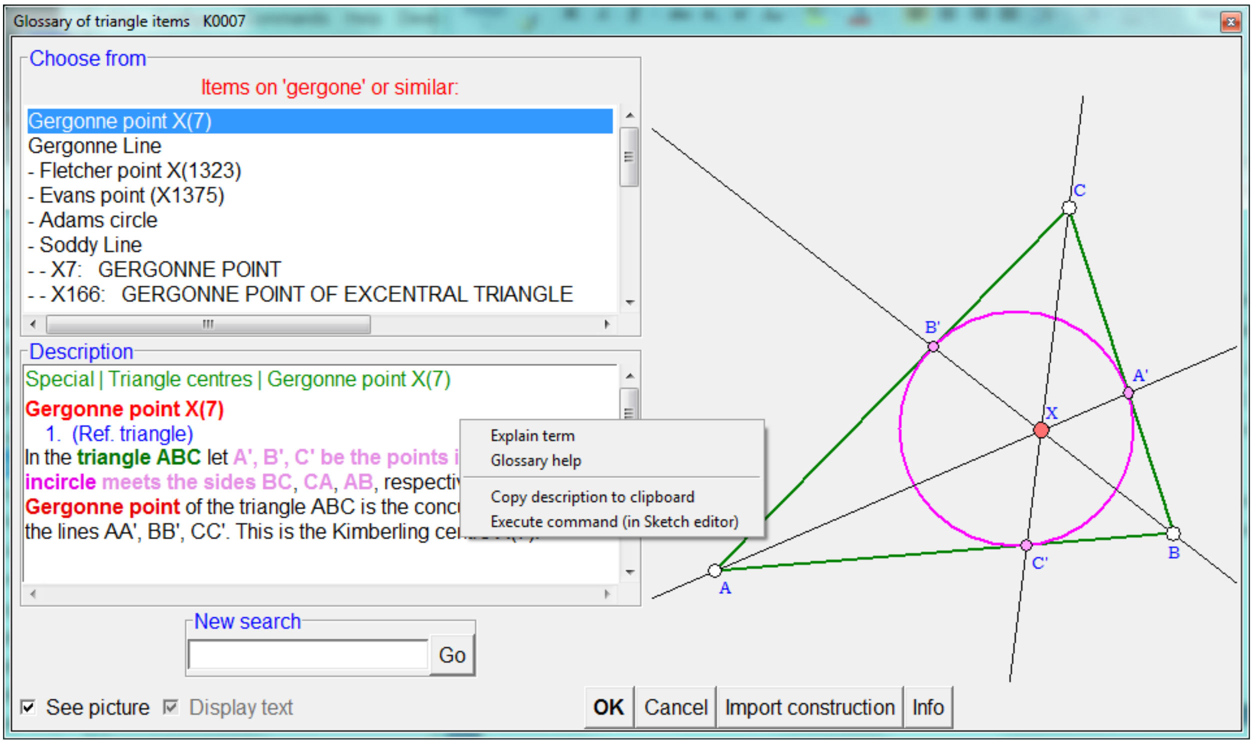


Figure 1

In the form there is a **description** and **illustration** of the sought term. Since there is a command related to the Gergonne point, the description contains also the position of the command in the menu (first line) and the arguments of the command.

A right click on the description provides a menu with more help.

Explain terms turns the cursor into a question mark with arrow. A click of the arrow at some word in the description (for example 'incircle') gives an explanation of the meaning of the word (possibly including the adjacent words).

Glossary help turns the cursor into a spider. A click of the spider at some word in the description activates the glossary for the pointed word.

Execute command is active only in Sketch editor and activates the form containing the desired command. In our case to the list of triangle centres where we can select the Gergonne point.

1.2. Triangle commands

Here is a list of all triangle commands. We can view them separately according to the type of objects. The provides description has the same functionality as the description in glossary.

3. Simple observation of triangle objects

Assume a construction has been imported from some dynamic geometry system or constructed in OK Geometry. We describe here how to relate constructed objects to a given triangle – called reference triangle.

We explain this on a simple example (Figure 2). Let A’, B’, C’ be the basepoints of altitudes from vertices A, B, C of triangle ΔABC. Furthermore, let A’’, B’’, C’’ be the midpoints of B’C’, C’A’, and A’B’. By observing the configuration or in some other way, we find that (apparently) the lines AA’’, BB’’, CC’’ meet in a common point P. We wonder how is this point related to ΔABC.

If we are not in Plus mode, we need to switch to it – in the menu on top choose the command *Configure|Working mode|Plus* .

Then click on the command  *(Analyse objects in reference triangle)* on the taskbar on right.

Then click on the objects you want to know about, in our case the point P.

If the reference triangle has not yet been set, a form appears, in which you write the labels that determine the reference triangle (Figure 3). The reference triangle is shown in pale yellow. You can always declare the reference triangle with the command *Comands|Set reference triangle* .

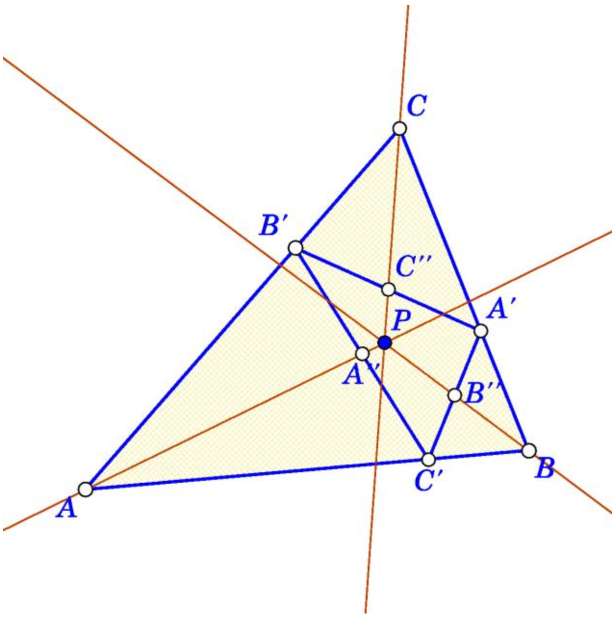


Figure 2

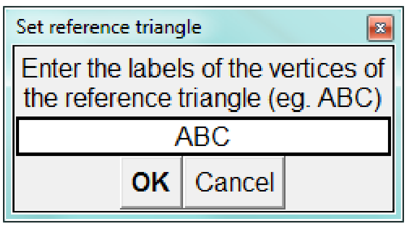


Figure 3

After a short calculation a form appears with the observational results ().

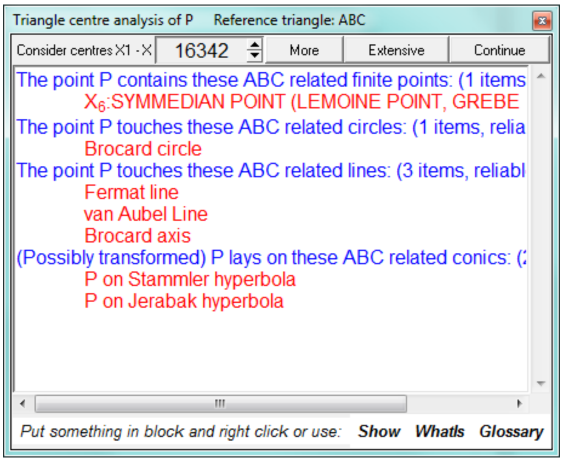


Figure 4

The form claims that the point P is the symmedian point (commonly denoted as X6) and presents some additional information regarding the point P. To get more information, click the *More* or *Extensive* button.

The objects in the presented form can be readily displayed. For example, to display the Fermat line, put the text Fermat line in block (or just position the cursor on the text) and click the ***Show*** command – the Fermat line appears on the construction (Figure 5).

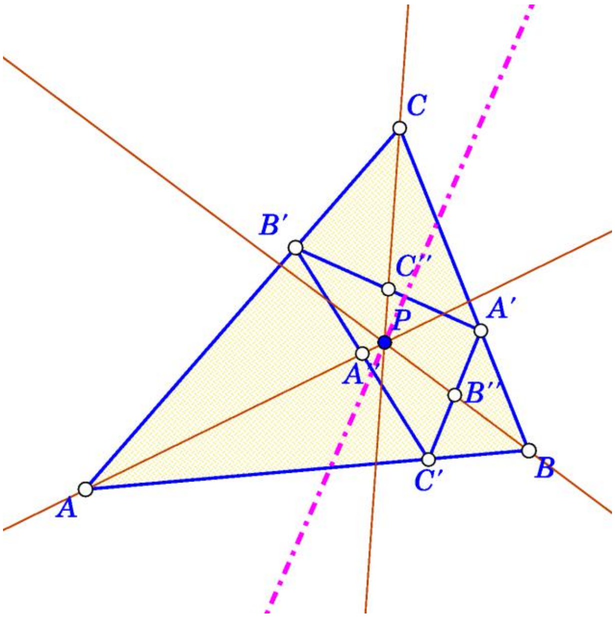


Figure 5

In a similar fashion we can obtain an explanation of the terms in the form. For example, to obtain some basic information of the Fermat line (or some other term), put the text Fermat line in block (or just position the cursor (question mark with arrow) on the text) and click the ***WhatIs*** command – the Fermat line appears on the construction (Figure 6).

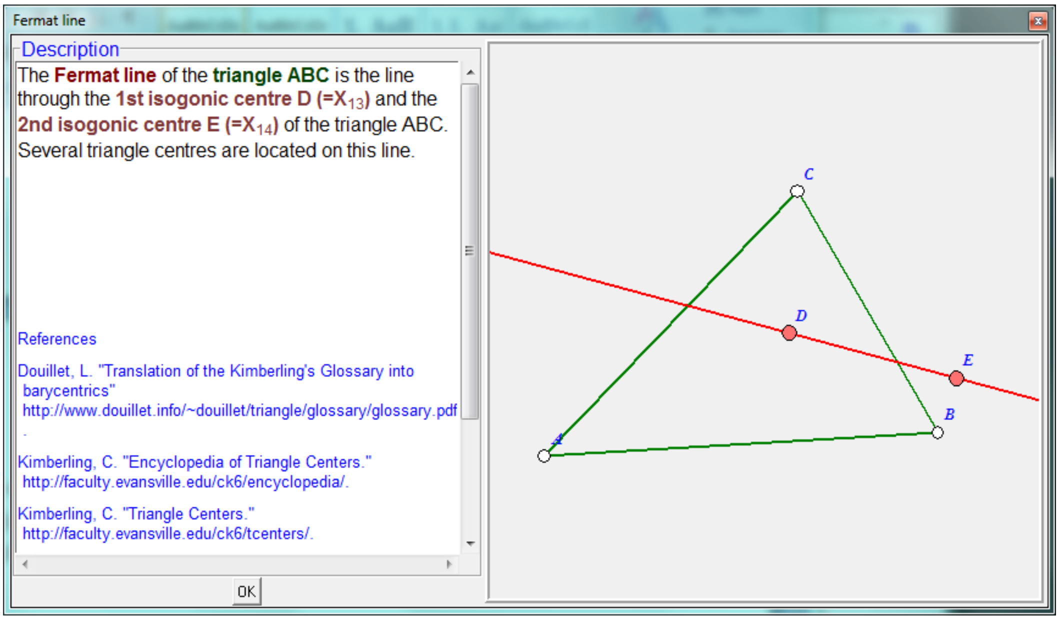


Figure 6

Finally , a glossary that relates a term to other similar term, used in triangle analysis, is provided. For example, to find objects related to symmedian point, put the text Symmedian point in block (or just position the cursor (spider) on the text) and click the ***Glossary*** command (Figure 7).

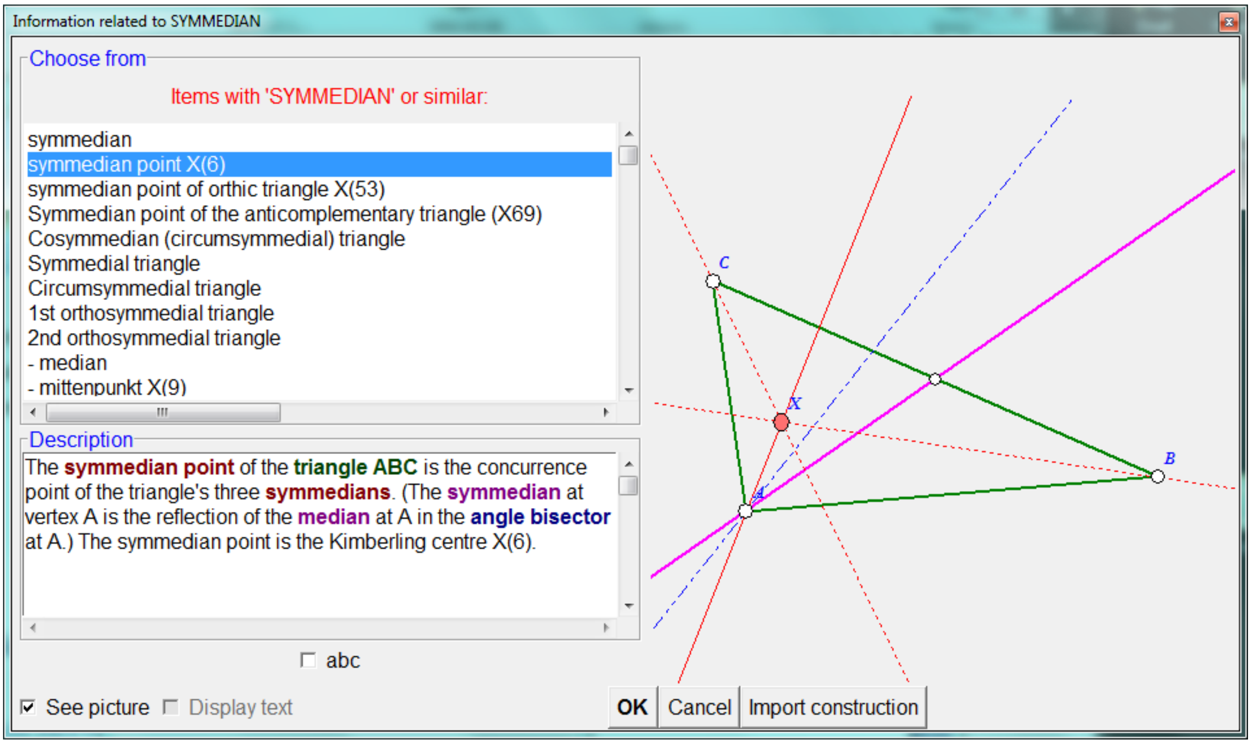


Figure 7

Some additional commands related to the form (Figure 4) are available by a right-click on the form.

In the form (Figure 4) we can also set the range of points (from the ETC list of triangle centres[1]) to be used in the analysis. Complete analysis can consider the about the first 16000 points, a somehow restricted and less reliable analysis can take account of more than 30000 points.

4. Special commands

In Plus mode of the Sketch editor there is an additional group of commands (**Special commands**). All commands in the group are specific for triangle geometry and most have a similar syntax.

The special commands comprise the construction of various objects related to a given triangle. The triangle we refer to is hereby called ***reference triangle***. Since a same triangle is often repeatedly used as a reference triangle, it is a common practice to declare a default reference triangle for most of the commands in the group. This is done with command *Commands|Set reference triangle* or *Special|Set reference triangle*. The commands of this group refer to the default reference triangle; however it is always possible to specify another reference triangle *as the first argument of the command*. Figure 8 shows a typical group of commands (characteristic objects of a triangle). Note the red line with checkmark at bottom, which states that the default reference triangle will be used, unless the checkmark is cleared. Note also the checkmark *See picture* at bottom. If this checkmark is on, then a pictorial representation of the object appears on the right.

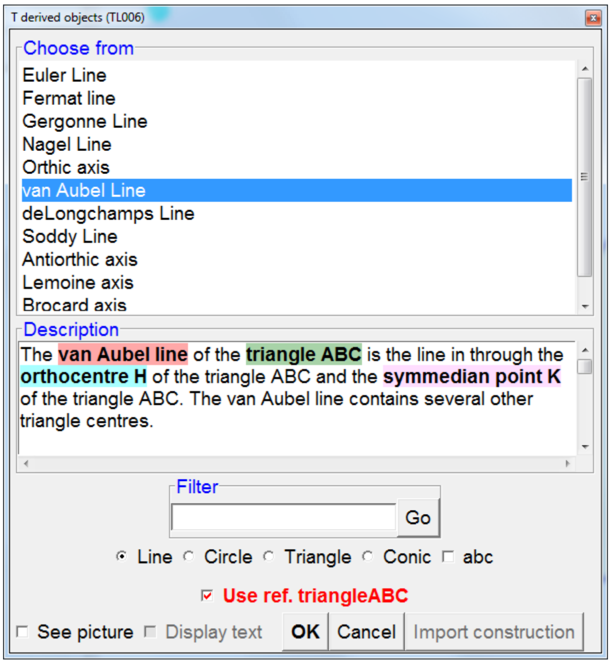
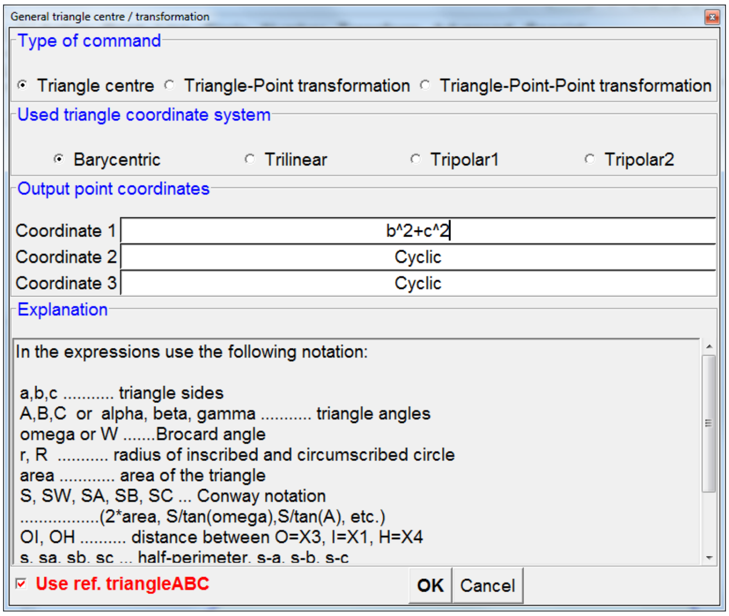


Figure 8

The Special commands menu the commands are grouped according to the input objects:

[1] ETC is acronym for Encyclopedia of Triangle Centers (https://faculty.evansville.edu/ck6/encyclopedia/ETC.html).

Triangle centres	Triangle centres are organised in five lists: - simple centres, i.e. the four classic triangle centres; - advanced centres, i.e. the most known centres, e.g. isodynamic point; - ETC centres, the first 16341 centres in ETC list of triangle centres; - ETC+ centres, the centres from 16342 on in ETC list of triangle centres; - bicentric, some well known triangle bicentres, e.g. Brocard points. A standard notation is used for the centres from the ETC list, i.e. X_n, X*n* or X(*n*) for the *n*-th centre. The centre X_0 has a special meaning – it is the variable centre, which can take place of any centre.
Triangle derived objects	The classic triangle related objects are organised in several lists: - triangle lines, e.g. the Euler line; - triangle circles, e.g. the Brocard circle; - triangles, e.g. the orthic triangle; - conics, e.g. the Kiepert hyperbola.
Triangle/Point derived objects	A given point determines various objects related in a given reference triangle. Such objects are organised in several lists, depending on the type of the resulting objects. The considered type of resulting objects are: - points, e.g. the isogonic conjugation of a point in a reference triangle; - lines, e.g. the trilinear polar of a point in a reference triangle; - triangle, e.g. the pedal triangle of a point in a reference triangle.
Triangle/Point/Point derived objects	This command contains a list of binary point operations that are defined in a reference triangle, e.g. cross-conjugation of two points in a reference triangle.
Objects by triangle centres	Given a reference triangle, it is possible to visualise triangle centres and to define and visualise objects passing through given triangle centres. In the form appearing form simply write the involved centres. Here are some examples: 1-100 The first 100 centres (2,14) The line through X(2) and X(14) (2,14,15) The circle through X(2),X(14),X(15) (5,6,7,8,9) The conic through X(5),X(6),X(7),X(8),X(9)
General triangle derived point	Here it is possible to analytically define centres, point transformations or point operations in a reference triangle. In the definition barycentric, trilinear or tripolar coordinates are allowed, as well as various triangle parameters. In the example below (Figure 9) in the reference triangle with sides *a,b,c* we

	set a point with barycentric coordinates (b^2+c^2,c^2+a^2,a^2+b^2).  **Figure 9**
Various constructions	Here are collected various constructions of points, lines, circles, and conics with various input objects. We point only some remarkable examples: - perspectivity centre of two triangles (if it exists); - orthology centre of two triangles (if it exists); - parallelogy centre of two triangles (if it exists); - cyclology centre of two triangles (if it exists); - trilinear pole of a line wrt. A triangle; - perspective axis of two triangles (if it exists); - the perspeconic conic of two perspective triangle. The commands in this group do not make use of the default reference triangle.
Set reference triangle	This command is used to set the reference triangle to be (optionally) used in other special commands. The vertices of the reference triangle should be labelled. You can specify the reference triangle by writing the labels in the entry that appears or by clicking on the vertices of the reference triangle. To clear the reference triangle enter in the form an empty line.
A triangle	This command is active only at the very beginning of the construction. It generates a triangle with labelled vertices and cyclic structure.

Declare cyclic objects	Declare three objects (of the same type) as cyclic in accordance to the current cyclic construction. When the Sketch editor is set as 'cyclic construction' (see the command below), any command that involves one or more cyclic objects is repeated cyclically on the related triads of objects.
Cyclic construction	Turns the Cyclic construction ON and OFF. When ON, the display pointer assumes the form similar to circle and the commands are executed cyclically relative to the reference triangle. **Use with caution.**
Analyse object in reference triangle	The command provides information about the triangle centres that lay on a given object. Just pick a point, a line, a circle or a conic. If a reference triangle was not set previously, you first specify the reference triangle. After a while, you obtain the observed properties that involve triangle centres and the common objects related to the reference triangle. You may obtain also a longer or an extensive list of observations. To inspect other objects click **Continue** and pick the next object to be analysed. Here are some illustrative observations: - The picked point is the centre X(42) of the reference triangle. - The picked line is tangent to the nine-point circle. - The picked point is the intersection of lines X(6)X(9) and X(5)X(7). - The picked line passes through X(21) and the isogonal conjugate of X(35). **Note.** This command is intended for a quick analysis. For a more elaborated analysis, see Section 5. **Note.** Right click on the displayed results: there are several available commands for visualising the mentioned objects and obtaining information about the mentioned objects (place the cursor set a block in the text appropriately).

All commands for triangle centres and other objects, related to a triangle, are applied in a similar way. In the form that appears (Figure 10) first specify the type of the resulting object and then select the operation. Suppose we look for the circumcevian triangle of a given point with respect to the reference triangle (Figure 10). Since our object is defined by the reference triangle and one point, choose the **Triangle/Point derived objects** command. In the form select first the 'triangle' option (since the resulting object is a triangle). Among the listed operations select the Circumcevian one. ***Pay attention to the left bottom option: In case a reference triangle was set previously, specify whether you want to use the previously specified reference triangle or to some other triangle.*** As you can see, the form contains a short description of the operation with optional illustration (option 'See').

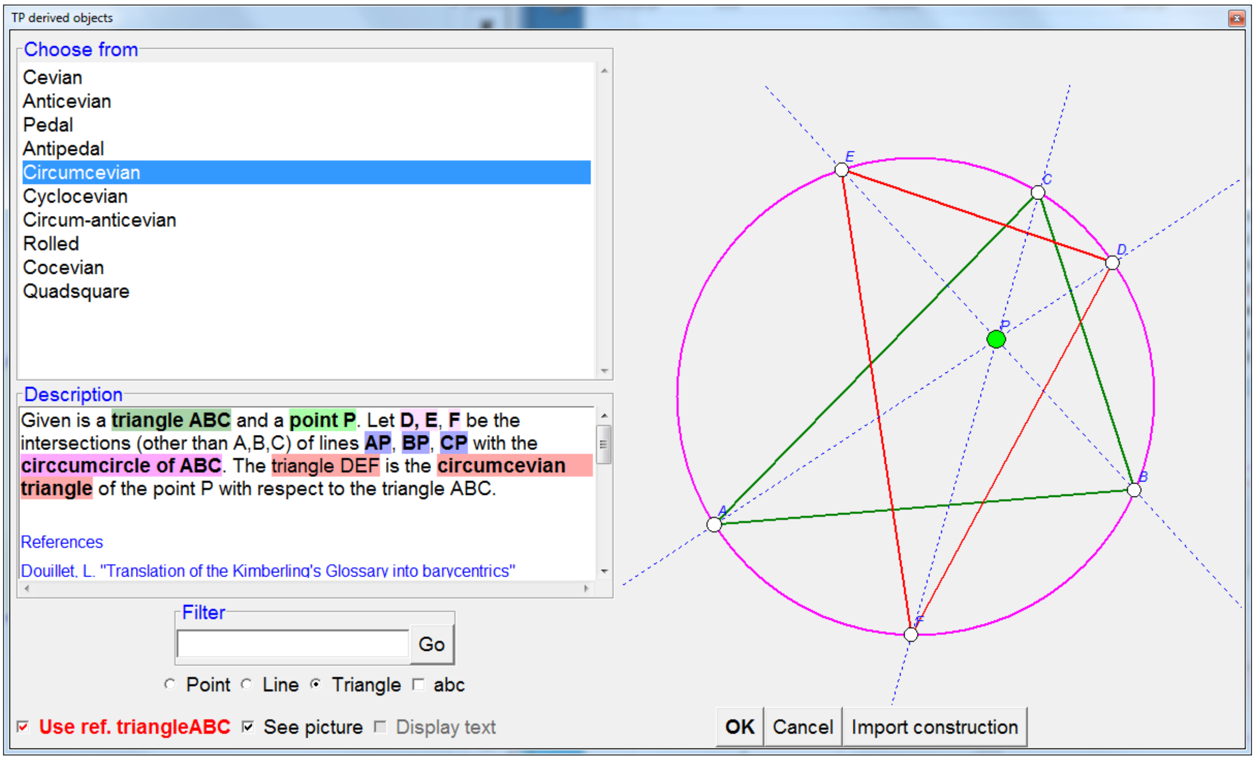


Figure 10

Here is a detailed description of the steps to follow:

1. Select the type of the resulting object (in the case above choose among Point, Line, and Triangle). A list of available object/operations appears. Checkmark **'abc' option** if you want the list to be alphabetically ordered.
2. On the displayed list click the object or operation you are interested in. A short description of the object/operation appears. If you want an illustration of the description, checkmark the **'See' option**.
3. In case a reference triangle was previously defined, there is the option for using it as the reference triangle for the operation.
4. Click OK and select the involved objects:

*If 'Use ref. triangle' is **ON***	*If 'Use ref. triangle' is **OFF** or is not present*
1. Since the reference triangle is set, you need only to pick the additional arguments (depending on the command can be none, one point or two points).	1. ALWAYS specify first the reference triangle. Either pick its three vertices OR pick the polyline consisting of the triangle's sides. 2. Pick the additional arguments (depending on the command can be none, one point or two points).

5. Cyclic constructions

Cyclic constructions allow effective constructions in a triangle. They are applied when a construction is repeated in the same fashion for all three sides (vertices, etc.) of a triangle. We illustrate the way of carrying out cyclic construction on a simple example, the Napoleon point of a triangle.

The (outer) Napoleon point of the triangle ABC is constructed as follows: on each side of the triangle place an equilateral triangle. The line segments connecting the three vertices with the centres of the equilateral triangles on the opposite sides meet in a common point, the (outer) Napoleon point of ABC.

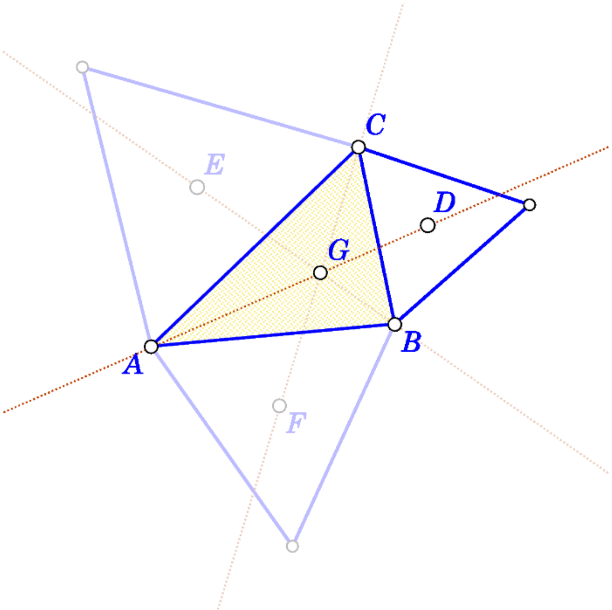


Figure 11

Here is how to do the construction as cyclic construction.

<table>
<tr><th>Step/Command</th><th colspan="2">Comment</th></tr>
<tr><td>Advanced|Shapes|Triangle</td><td>Construct a triangle.</td><td rowspan="5">All this can be done with single command Special|A triangle.</td></tr>
<tr><td>Actions|Label|Auto label</td><td>Label the triangle's vertices as A, B, C.</td></tr>
<tr><td>Special|Reference triangle</td><td>Declare the triangle ABC as reference triangle (which becomes yellowish).</td></tr>
<tr><td>Special|Cyclic objects</td><td>Declare the objects (points) A, B, C as cyclic (wrp. to the reference triangle).</td></tr>
<tr><td>Special|Cyclic polyline</td><td>Declare the polyline ABCA as cyclic. (in this way also the sides of the reference triangle are declared as cyclic).</td></tr>
</table>

Special|Cyclic	Cyclic construction is turned ON.
Advanced|Shape| Equilateral triangle	Construct the equilateral triangle on the side CB. Simultaneously, bleached equilateral triangles are automatically constructed on the other two sides of the triangle ABC.
Special|Triangle centres|incentre	Construct the incentre D of the equilateral triangle with CB as side. Incentres of the other two equilateral triangles are automatically constructed.
Actions|Label|Auto label	Label the remaining centres of equilateral triangle as E,F. Note. Labelling vertices is not treated as cyclic operation.
Line|Line segment	Construct the segment AG. Simultaneously two other cyclically defined segments are constructed.
Special|Cyclic	The Cyclic construction is turned OFF. This is necessary because of the command that follows.
Point|Intersection	Construct the intersection point J of segments AG and BH. Turning the Cyclic construction OFF was necessary, otherwise three intersection points (at the same place) would be constructed.

6. Triangle analysis module

The Triangle analysis module is an advanced observational tool for triangle related objects. Often one wanders, in what way a given point, line, circle or conic is related to a given triangle. In such cases triangle analysis may provide reliable hypotheses. The analysis provides reliable indications about which ETC centres or certain transformation of ETC centres lay on a given line or conic and how given points, lines, circles relate to many of the common objects of the reference triangle.

Suppose that during the study of a reference triangle ABC we come across points P, Q, R. For some reason, we would like to know something more about these points and about the line PQ. Using the **Triangle analysis**, OK Geometry might, for example, provide reliable observations:

- that P is the centre X_{541} of ABC, or that P is the cyclocevian conjugation of X_{541} in ABC, or that the complement of P in ABC is X_{541}, or that the Ceva Q conjugate of P in ABC is X_{541},
- that P is the intersection of lines $X_{17}X_{541}$ and $X_{16}X_{125}$,
- that P lays on the Euler line, or that the isogonal conjugation of P lays on the Kiepert hyperbola,
- that the line PQ is tangent to the Spieker circle of ABC, or that P is the inverse of X_{17} in the nine-point circle of ABC,

- that the circle through P,Q,R contains X_{541} and the isogonal conjugation of X_{130} in ABC.

Triangle analysis can also compare triangle centres of two triangles, i.e. provide a list of all coincident centres of two given triangles. For example, that it is very likely that the centre X_{541} of ABC is the same point as X_{54} of PQR.

The use of triangle analysis module is rather straightforward: Given a construction, we just specify which objects to analyse with respect to which reference triangles.

3.1. Accessing the Triangle analysis module

To access the Triangle analysis module click the triangle (Δ) symbol in the main menu bar. Note that for a given construction the analysis can be performed only on objects that can be identified with labelled points on them. In particular, the vertices of the reference triangle(s) should be labelled, and the points, lines, circles, conics to be analysed should respectively contain 1,2,3,5 labelled points.

3.2. Triangle analysis

The best way to learn how to use triangle analysis is by using it. For this purpose some simple and advanced examples are provided (see Section 7). You can learn how to use the triangle analysis module by working out these examples step by step. However, for the sake of completeness we provide here a description of the form that is used in the module.

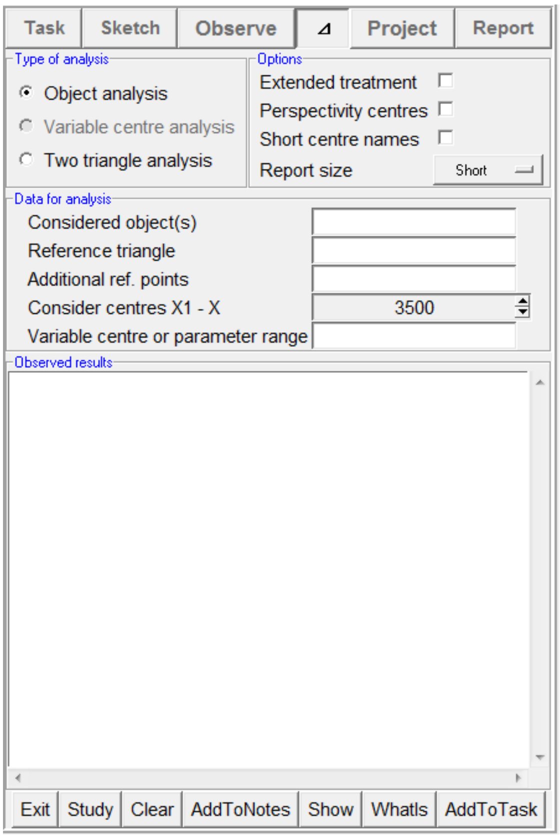


Figure 12

The description below refers to Figure 12. This kind of form appears on the left-side pane during the Triangle analysis. Note that depending on the construction some of the fields may be missing or may be disabled. Here is a brief description of the meaning of the entries:

Entry	**Description**
Considered objects	In the entry write objects you want to consider in the analysis. Separate the objects by a space or comma. Refer to points as they are labelled, e.g. A' or B or C3, refer to lines by specifying two points on them, e.g. A'B, and refer to circles by specifying three points on their circumference, e.g. A'BC3. Refer to conics by specifying 5 points on them, e.g. A'BC3DE.
Reference triangle	Specify the reference triangle by naming its vertices, e.g. ABC. All the centres and triangle object in the analysis will refer to the reference triangle. If two or more reference triangles are specified (separated by spaces or commas, e.g. ABC D1E'F), the considered objects will be analysed with respect to each of the specified reference triangles. **Note. Avoid studying triangles that are isosceles or near-to-isosceles**. In such triangles the density of centres becomes prohibitively high, so that OK Geometry refuses to analyse the centres.
Additional reference points	In the analysis the position of a considered object is compared to the position of the reference points and to the characteristic objects of the reference triangle. By default the reference points are the centres of the reference triangle and, eventually, their transformations. If we want to consider additional reference points (besides the centres of the reference triangle) we specify them here.
Condition parameter	This entry appears only if there is some check in the construction. There are situations when we want to study an object only when a condition is satisfied (e.g. that three lines are collinear). Such situations occur, for example, when the studied construction is based - on an implicit construction, or - on optimisation of a parameter, or - on a variable triangle centre X0. In such cases, i.e. when we want to disregard the failed instances, we select here the condition which indicates the success of an instance of the construction.

Considered centres	Here we specify the highest index of ETC triangle centres to be considered in the analysis. Note that the when maximal index is chosen the analysis is les reliable and not all tests are performed. By reducing the number of considered centres we avoid extensive lists of results containing less known centres. Reducing the number of centres, obviously, makes the analysis faster.
Variable centre range X	This entry appears only if the construction contains the variable centre X0, which can take the place of any ETC centre. In the entry we specify, which centres to consider in this sense. For example, the entry 1-10, 16,17 leads to the analysis of 12 cases in which X0 takes respectively the place of X1, X2,..., X10, X16, X17.
Extended tratment	Checkmark this option if you are interested not only on properties of the considered points in relation to the reference points (i.e. triangle centres) but also of various transformations of these points. For example, suppose that we consider a point, which has the property that its isogonal conjugation lays on the Euler line of the reference triangle or that the considered point is the isogonal conjugation of the Adhara centre (X_{911}) . If the **Extended treatment** is check-marked, then such properties will be spotted, otherwise not.
Perspectivity centres	This option is valid only when a triangle is analysed in relation to the reference triangle. The analysis always looks for reference triangle related triangles that are in some way perspective to analysed triangle. When this option is ON, al the perspectivity centres are calculated and stored, so that they can be later accessed and studied. For example, let A'B'C' be the extouch triangle of the reference triangle ABC. If the analysis is performed with the Perspectivity centres option ON, after the Study command is pressed, OK geometry reports the observation of 39+92 perspectivities of A'B'C' and various triangles related to ABC. Some of these cases may coincide. These perspectivities give rise to 198 centres (not all different among them) that are denoted by %1 to %198. To state a case of perspectivity, the report says that A'B'C' is related to Garcia inner triangle in the following way: perspective (%59), orthologic (%60, %61). Thus %59 is the perspectivity centre of ABC and A'B'C', the point %60 is the orthological centre of ABC and A'B'C' and the point %61 is the orthological centre of A'B'C' and ABC. To find more about the point %60 simply write %60 as the considered object and study it. The notation %% encompasses all points %1-198 without repetition.
Short centre names	When this entry is check-marked the descriptive names of the centres are omitted in the report.

Report size	When this entry is check-marked the report contains only a first few items of each found property. Example. A considered point may lay on tens of lines passing through two or more ETC centres of the reference triangle. Each of these lines may contain tens of ETC centres. If **Short report** is check-marked, then the report will contain only a few of these lines with only a few centres on each line.

3.3. Object analysis

In object analysis OK Geometry observes the relations between objects in the construction and the centres of the specified reference triangle as well as with several standard objects related to the reference triangle.

Be sure that the **Type of analysis** is set to **Object analysis**. When the data for analysis are properly filled the analysis can begin. During the analysis you use the buttons in the bottom, many commands are available in the context menu (right click on the Observed results section). Here is a brief description of what you can do:

1. To execute the analysis press the **Study button.** A report will appear in the Observed results section. An illustrative example of the beginning of a report (about some point D) is shown in Figure 13.

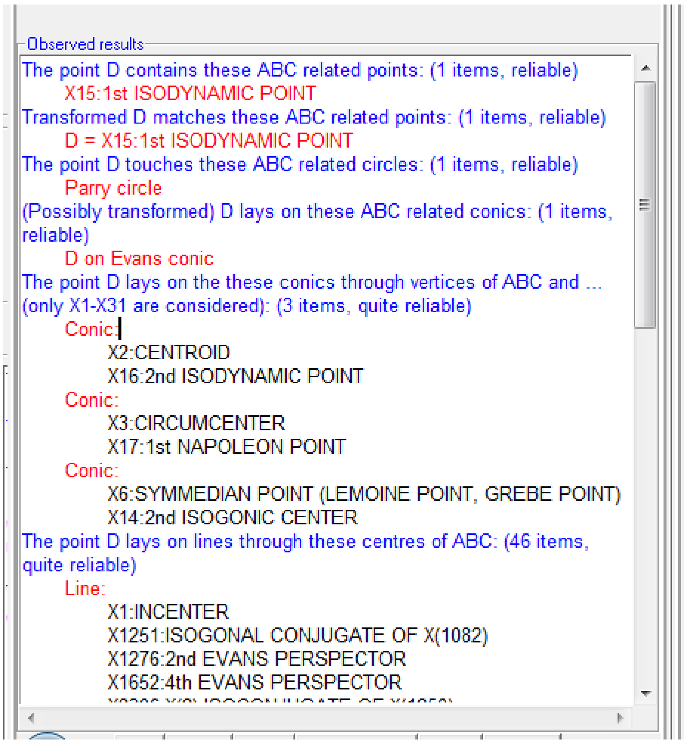


Figure 13

2. The analysis can be performed several times in sequence. Just change the data and options and press the **Study button**.
3. Reports are always appended to the previous content of the Observed results section. At any time you can clear the content of the Observed results section with the **Clear** button.
4. Reports can be freely edited.

5. If you want to include a piece of the Observed results in the OK Geometry report (i.e. the printable report) for the studied construction, put that part in a block and press the **AddToReport button**.
6. You will often want to clear the Observed results section before a new Study analysis. Use the **Clear button** to delete the Observed results section.
7. To obtain some explanation about objects that are mentioned in the Observed results section, just position the cursor on the enquired term (or put the terms in a block) and press the **WhatIs button**. For example, to find out the definition of the Parry circle or of X_{17}, put the cursor on 'Parry' or on 'X_{17}' and press **WhatIs button.**
8. It is possible also to visualise the objects that are mentioned in the Observed results section. Just position the cursor on the name of the object to be shown or put in a block the objects to be shown, and press the **Show button**. For example, to visualise X_{16}, position the cursor on X_{16} and press the **Show button**. To visualise the conic through X_2, X_{16}, and the vertices of the reference triangle, put in block the three lines from Conic to X_{16}, and press the **Show button**.
9. You may want to add the visualised object to the construction (so that they may be used outside the analysis module). To do this right click on the Displayed results section and select the **Set as construction** command. For example, to add to your construction the Parry circle (of the reference triangle):
   - First put the Parry circle in block and press the button **Show**. The circle appears.
   - Right click the mouse and select the **Set as construction** command (the mouse cursor should be on the Observed results or on the construction pane).
   - In the form that appears choose the attributes of line(s) to be added and click **OK**.
10. Finally, to exit from the Analysis module, press the **Exit button** or press again the (Δ) button in the main menu bar.

3.4. Variable centre analysis

This is a quick and restricted version of the analysis for constructed points that depend on the variable centre X_0. In the analysis X_0 takes the position of various ETC centres. For each such case (position) the analysis looks for matches of the constructed point (or its transformation) with some ETC centre of the reference triangle.

For example, consider the midpoint of a centre and its isogonal conjugation. With the variable centre analysis OK Geometry observes for all centres of the reference triangle the mentioned midpoint matches another cntre of the reference triangle.

3.5. Comparing the centres of triangles

In this analysis OK Geometry looks for matches of the (possibly transformed) ETC centres of the reference triangle and the ETC centres of another triangle (=considered object).

7. Examples

4.1. The Privalov conic

It is well known that in a given triangle ABC the vertices of the cevian triangles of any two different points D, E (not laying on the sidelines of ABC) are conconical (Figure 14 left). We shall consider the case of two special points: the Gergonne point (D) and the Nagel point (E) of a given triangle ABC. The Gergonne point od ABC is the point of concurrence of the line segments connecting the points where incircle of ABC touches the sides with the opposite vertices. Similarly, the Nagel point of ABC is the point of concurrence of line segments connecting the points where the excircles touch the sides of ABC with the opposite vertices.

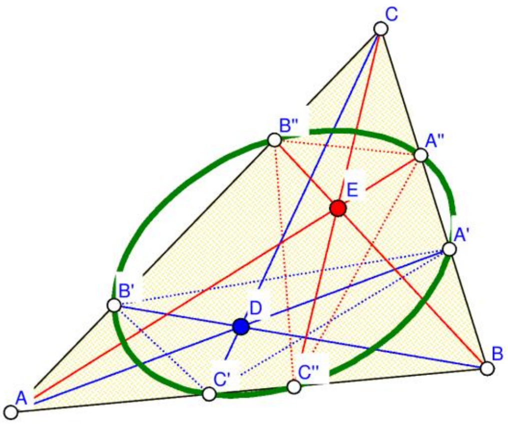


Figure 14

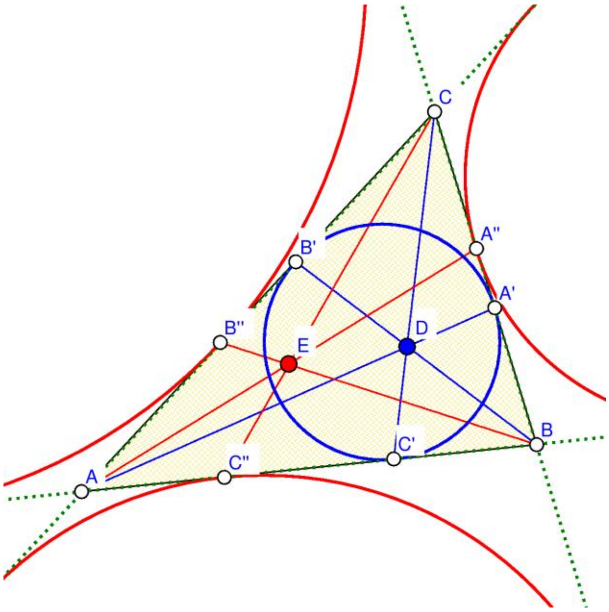


Figure 15

Here is a quick and efficient way of executing this construction:

Steps	Commands	Comment
1. Draw a triangle ABC	***Point|Point*** (for A,B,C) ***Line|Polyline*** (to connect A-B-C-A) ***Special|Reference triangle***	
2. Position the Gergonne point D and the Nagel point E of ABC	***Special|Triangle centres|Advanced|Gergonne point*** ***Special|Triangle centres|Advanced|Nagel point***	Keep the check-mark for using the reference triangle, since we construct the Gergonne and Nagel points of a previously defined reference triangle ABC.
3. Construct the Ceva-	***Special|Triangle point derived***	

triangles of D and E	***objects|Triangle|Cevian*** (choose then points D and E)	
4. Construct the ellipse and the characteristic points of the ellipse	***Circle|Conic 5 pts*** ***Point|Conic points***	Label the centre of the ellipse as S. (See Figure 16.)
5. Inspect the centre S of the ellipse	***Special|Centre analysis*** (pick S)	OK Geometry observes that S is the ETC centre X5452 . (See Figure 17.) A more detailed inspection with Triangle analysis suggest that, for example: S = Isogonal conjugate of the Cross conjugate of X6 and D, or S = Complement of the Isogonal conjugate of X1486, or S = Complement of the Isotomic conjugate of (X3434)
6. Inspect the points on the constructed ellipse	***Special|Centre analysis*** (pick the ellipse)	OK Geometry spots 3 centres along the ellipse (X3022, X3271, X4904). The extended search adds to this list 6 more transformed centres.

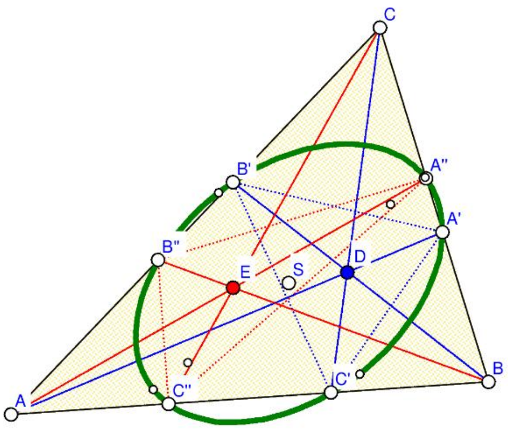


Figure 16

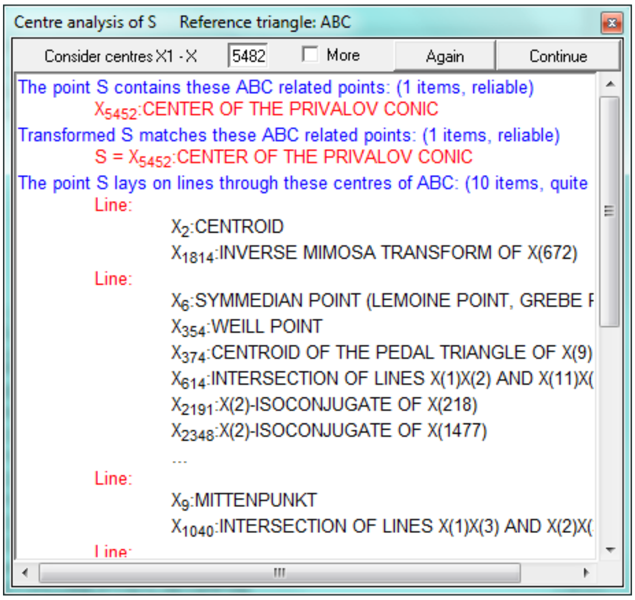


Figure 17

4.2. Circum-side-circle

Given is a triangle ABC. A circle having a side of ABC as its diameter is called a side-circle of ABC. Each triangle thus has three side-circles. We are looking for a way to construct the 'circumcircle' of the three side-circles of a given triangle, i.e. the three side-circles of ABC should touch from inside the constructed circle (Figure 18).

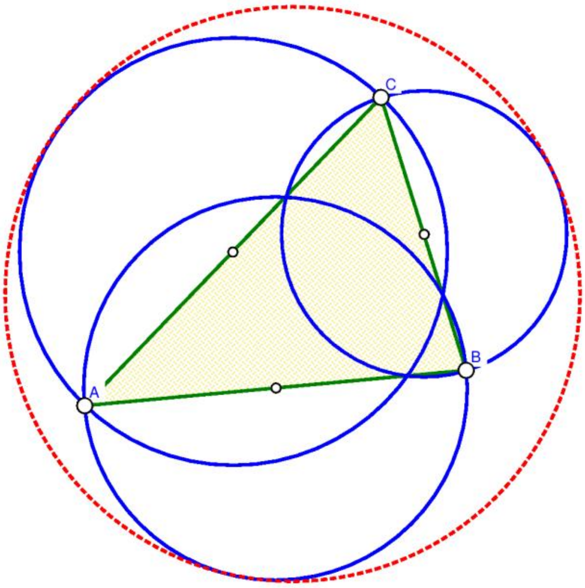


Figure 18

The strategy in the construction below is to find the three points of contact of the circum-by-circle with the three side-circles. Knowing how to construct (with Euclidean tools) these three points leads the way to the construction of the circum-by-circle.

Steps	Commands	Comment
1. Draw a triangle ABC	***Point|Point*** (for A,B,C) ***Line|Polyline*** (to connect A-B-C-A) ***Special|Reference triangle***	Try to make ABC visibly scalene.
2. Construct the three side-circles and the circum-side-circle	***Point|Mindpoint|***(pick ABC)***On each segment*** ***Circle|CircleC*** ***Circle|Circle 3 obj***	Use repeatedly the Alt button (in the editor's menu bar to obtain the desired circle that is tangent to three objects.
3. Construct the three points of tangency A', B', C' of the circum-	***Point|Point***	The **Observe** command reveals that AA', BB', CC' are concurrent

side-circle		in a point, say, P.
4. Construct the lines AA', BB', CC' and the concurrency point P.	***Line|Line 2 pts*** ***Point|Intersection***	
5. Inspect the properties of the point P in reference triangle ABC	Set ABC as the reference triangle. Enter the Triangle analysis module. Execute the Object analysis on the point P.	
6. Obtain information about the Paasche point	In the Triangle analysis module Position the cursor on X1123 and click the WhatIs button.	To obtain the basic information about the Paasche point position the cursor on this term and right click the **WhatIs** command (Figure 19).

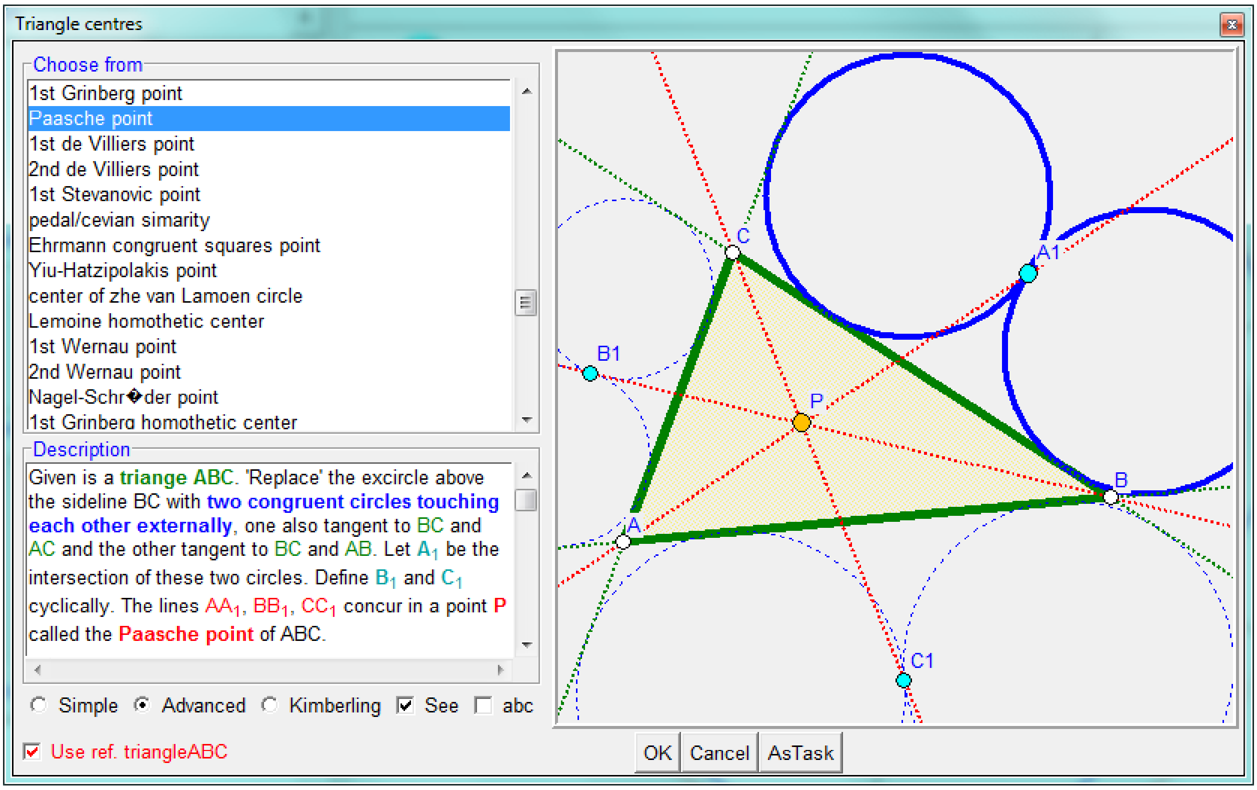
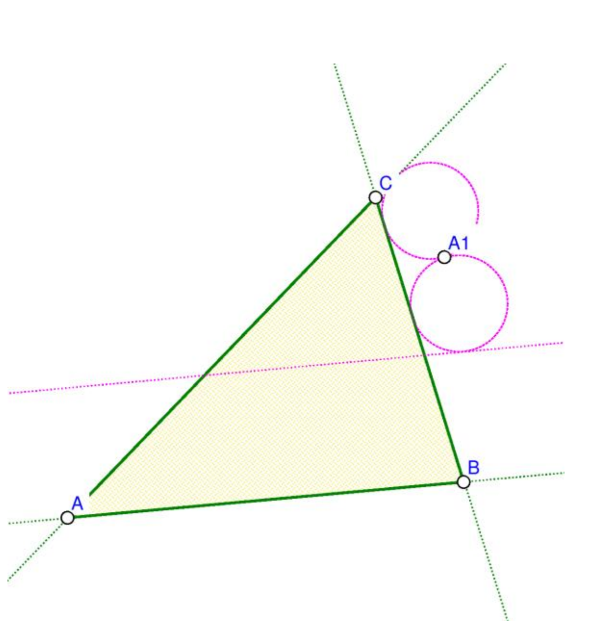


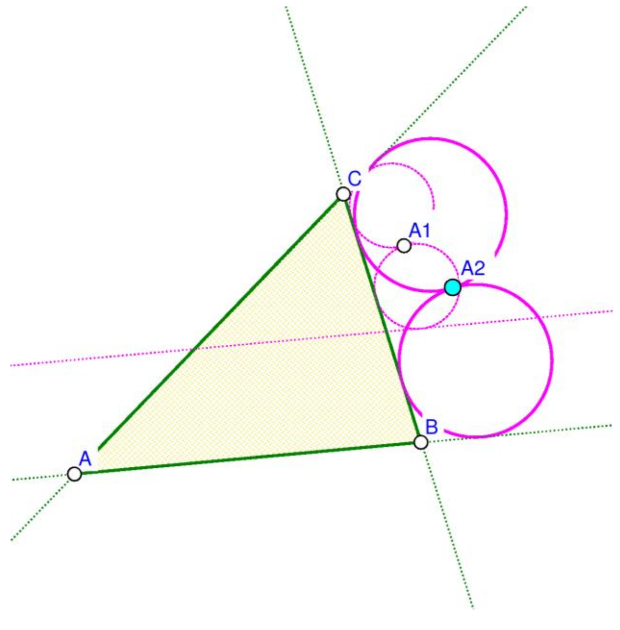
Figure 19

The Paasche point P of ABC can be constructed (using a homothety on a 'false' construction), and this gives way to the construction of the circum-side-circle. The construction steps are illustrated in Figure 20.

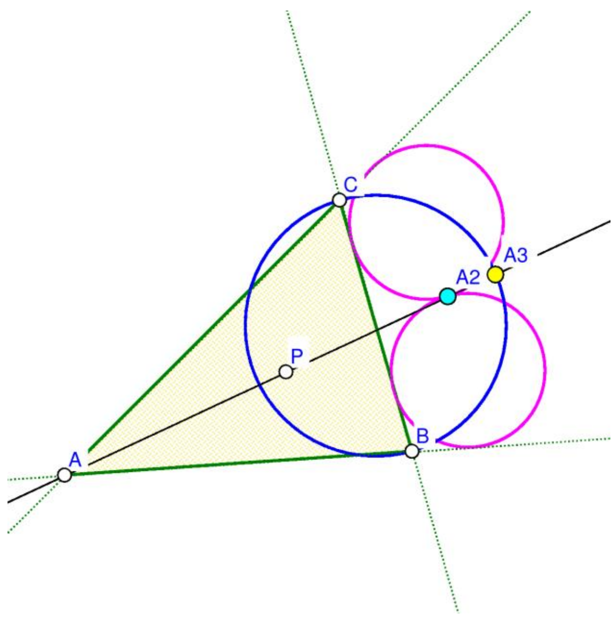
The correctness of the construction via the Paasche point can be verified with algebraic means.



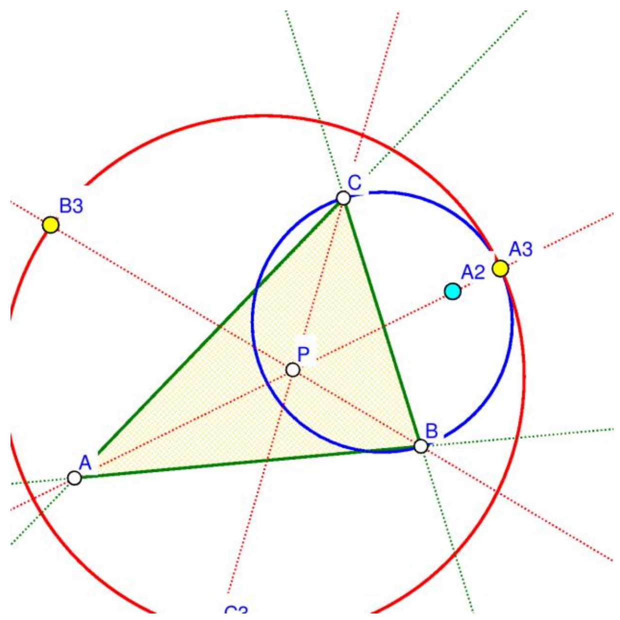
Step 1. Construct ‘false’ circles of equal radius (as shown), the first touching CA and BC, the second touching BC and the first circle.



Step 2. Using homothety obtain the tangency of the second circle and the line AB.



Step 3. Let A_2 be the common point of the two circles. Let A_3 be the intersection of AA_2 and the side-circle of BC. The point A_3 is the sought point of contact with the circum-side-circle of ABC.



Step 4. Obtain B_3 and C_3 circularly. The circum-side-circle is the circle through A_3, B_3, C_3. The common intersection of AA_3, BB_3, CC_3 is the Paasche point.

Figure 20

The circum-side-circle (Figure 18) is one of the 8 circles that are tangent to the three side-circles, see Figure 21. Here are some of the observed properties:

One of the 8 circles contains all three side-circles, this is the circum-side-circle with centre P0 – the complement of the equal detour point X(176).

Three of the 8 tangent circles are contained in exactly two of the side-circles, their centres are A_2, B_2, C_2 (see Figure 21). The lines AA_2, BB_2, CC_2 concur at point P_2 – the Yiu-Pasche point X(1659).

Three of the 8 circles are contained in exactly one of the side-circles, their centres are A_1, B_1, C_1 (see Figure 21). The lines AA_1, BB_1, CC_1 concur at point P_1 - the isogonal conjugate of the 3rd Kenmotu homothetic centre X(5414).

One of the 8 circles is contained all three side-circles. Its centre is P_3 – the complement of the isoperimetric point X(175).

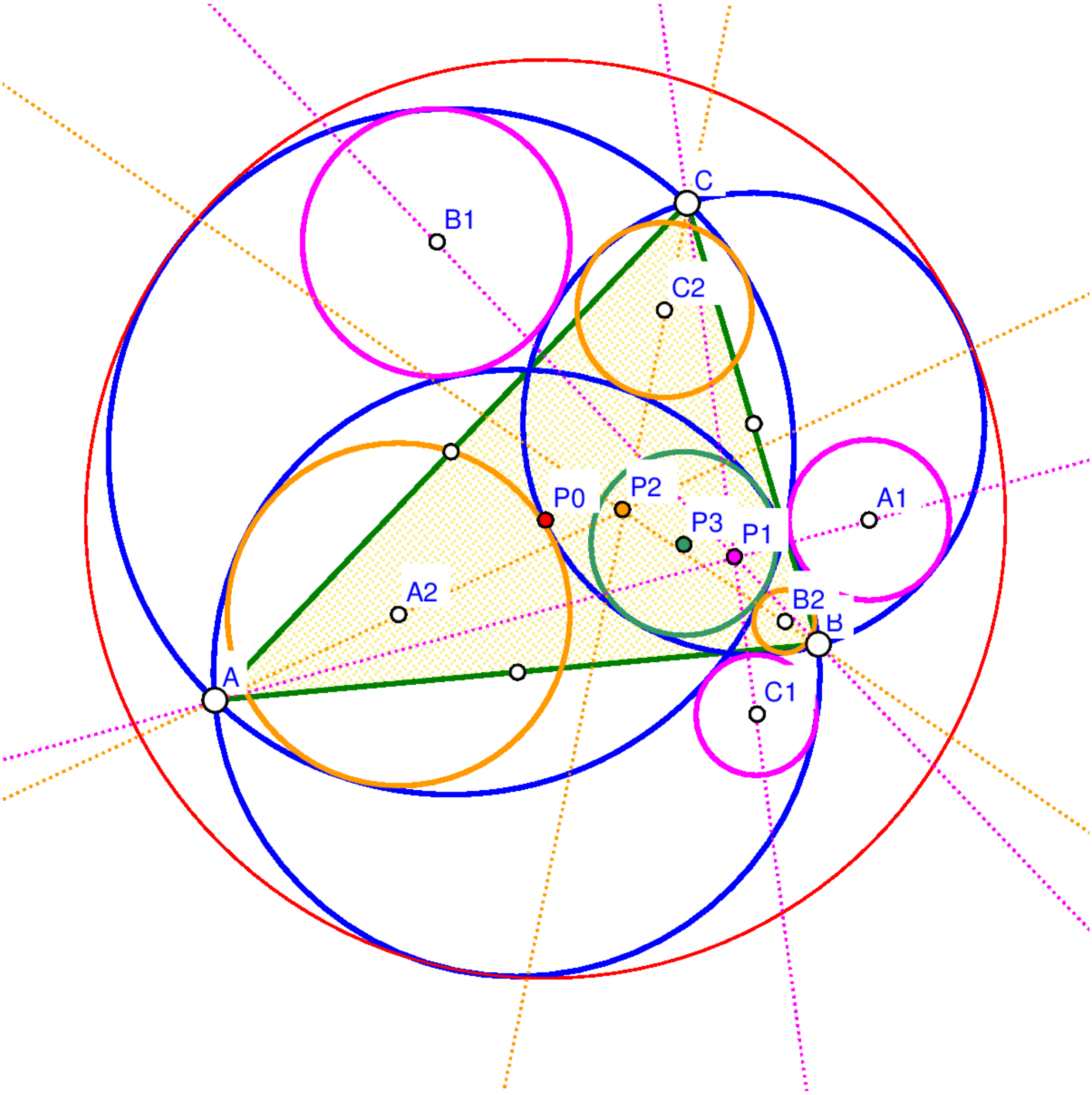


Figure 21

4.3. Trisecting triangle perimeter by pedal triangle

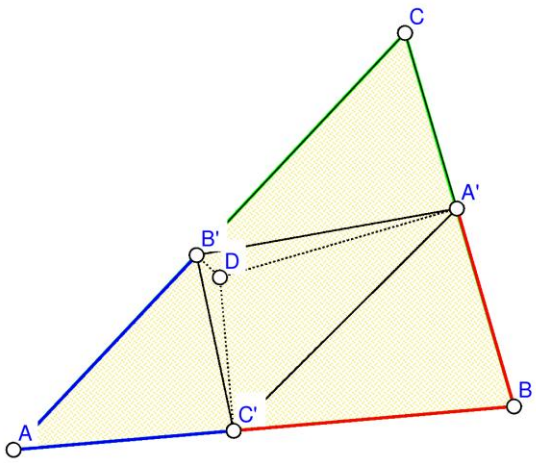


Figure 22

Given is a triangle ABC and a point D. The pedal triangle of D in ABC has as its vertices the projections A', B', C' of the point D onto the sidelines of ABC. We are looking for a point D with the property that A', B', and C' cut the perimeter of ABC into three pieces of equal length (Figure 22), so that |B'A|+|AC'| = |C'B|+|BA'| =|A'C|+|CB'|.

The steps below lead to a reliable hypothesis for the sought position of the point D. The strategy is to (1) construct a general triangle, a point D and the related pedal triangle, (2) redefine D with an implicit construction, (3) analyse the new position of the point D, (4) make an improved check for the suggested position.

Steps	Commands	Comment
1. Draw a triangle ABC and a point D	***Point|Point*** (for A,B,C) ***Line|Polyline*** (to connect A-B-C-A) ***Special|Reference triangle*** ***Point|Point*** (Construct the point D.)	Try to make ABC visibly scalene.
2. Construct the pedal triangle of D in ABC.	***Special | Triangle Point derived objects | Triangle | Pedal*** (Pick D.) ***Action | Lables | Label vertex*** (Label the vertices A', B', C'.)	Note that ABC was previously set as a referrence triangle.
3. Construct the obtained 'pieces of perimeter' as polylines and measure their length.	***Line | Polyline*** (Pick B',A,C'; repeat the command for C',B,A' and A',C,B'.) ***Number | Length circumference*** (Select the three constructed pieces.)	Name the lengths of the pieces as Len_A, Len_B, Len_C.
4. Check the sought condition	***Advanced | Check | Equivalence | SameValues3*** (Pick the values Len_A, Len_B, Len_C.)	Name the check as SameValues3. Obviously, it turns out as False.

5. Implicit (re)construction of D.	***Advanced | Implicit construction*** (Uncheck the restriction (OK); then pick the condition SameValues3 (OK) and the point D.)	The implicit construction should turn the condition to True.
6. Analyse the position of the point D	***Special | Centre analysis*** (Pick D.) OK observes that D is the centre X(165) – the centroid of the excentral triangle. To obtain a description of terms used in the result, position the cursor on it, right-click and use the command **WhatIs**.	Note that most items in the results are marked as 'unreliable', or even 'great caution'. This happens when the construction is based on optimisation or implicit construction.
7. Check the construction directly	***Action | Delete*** (pick the point D) ***Special | Triangle centres | ETC | X165*** Repeat the steps 2-4.	Note that the check is now much more accurate since there are no implicit points in the construction.

4.4. An analogy to Kosnita point construction

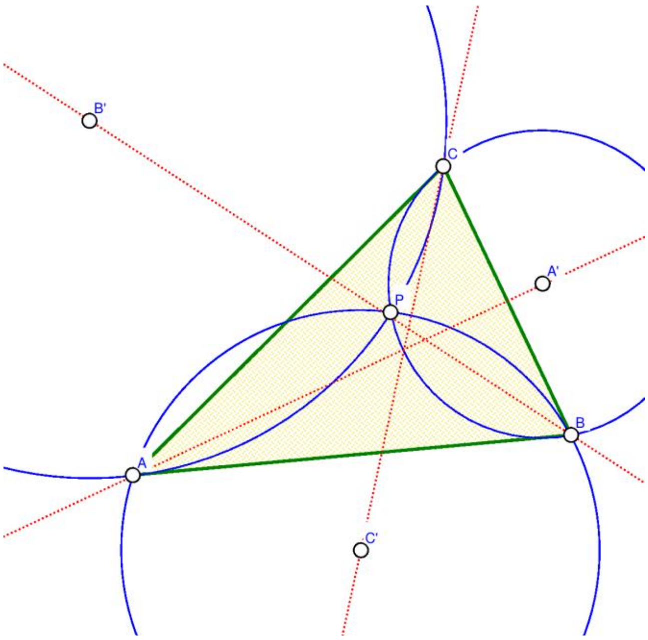


Figure 23

Given is a triangle ABC and a point P not laying on its sidelines. Let A' be the centre of the circumcircle of triangle PBC, and define B' and C' cyclically (Figure 23). In general, the lines AA', BB', and CC' do not concur. However, for certain points P this is the case. For example, when P is the incentre (X1) of ABC the mentioned lines concur in P. The lines concur also if P is the circumcentre of ABC – in this case the point of concurrence is the Kosnita point (X54) of ABC. Let us inspect for which triangle centres the lines AA', BB', CC' concur.

The strategy: (1) We shall construct a scalene triangle ABC. The point P will be set as a variable ETC centre. (2) Using the Variable centre analysis OK Geometry will produce hypotheses for which points the lines AA', BB', CC' concur, and what is the point of concurrence Q.

Steps	Commands	Comment
1. Draw a triangle ABC.	***Point|Point*** (for A,B,C) ***Line|Polyline*** (to connect A-B-C-A) ***Special|Reference triangle*** (ABC)	Try to make ABC visibly scalene.
2. Construct the variable centre P. Set P to be X(2).	***Special | Triangle centres | ETC | X0: VARIABLE CENTRE***	Note the centre indicator that appears on the left pane. Set the indicator to 2 (so that P and Q will not coincide) and press **Go**.
3. Construct the three circles and their centres A',B',C'.	***Circle | Circle 3obj*** (Pick P,B,C etc.) ***Point | Circle centre*** (Pick the three circles.) ***Label | Label vertex*** (Label the vertices as A', B', C')	
4. Construct the lines AA', BB', CC' and theri eventual point of concurrence Q.	***Line | Line 2 pts*** (Pick A,A', etc.) **Point | Intersection** (Let Q be the intersection of AA' and BB'). ***Advanced | Check property | Position | Concurrent lines*** (Pick the lines AA', BB', CC'. Name the condition of concurrency as ConcurrentL_1.)	Note that the Q is the point of concurrency only if ConcurrentL_1 is True.
5. Analyse the point Q for various centres.	In the Analysis module fill the entries as shown in Figure 24. After you press **Study** the observational results appear.	
6. B',C'.	***Circle | Circle 3obj*** (Pick P,B,C etc.) ***Point | Circle centre*** (Pick the three circles.) ***Label | Label vertex*** (Label the vertices as A', B', C')	
7. Construct the lines AA', BB', CC' and theri eventual point of concurrence Q.	***Line | Line 2 pts*** (Pick A,A', etc.) **Point | Intersection** (Let Q be the intersection of AA' and BB'). ***Advanced | Check property | Position | Concurrent lines*** (Pick the lines AA', BB', CC'. Name the condition of	Note that the Q is the point of concurrency only if ConcurrentL_1 is True.

	concurrency as ConcurrentL_1.)	
8. Analyse the point Q for various centres.	In the Analysis module fill the entries as shown in Figure 24. After you press **Study** the observational results appear.	

Ok Geometry observes that the concurrency of lines occurs for 285 of the considered centres.

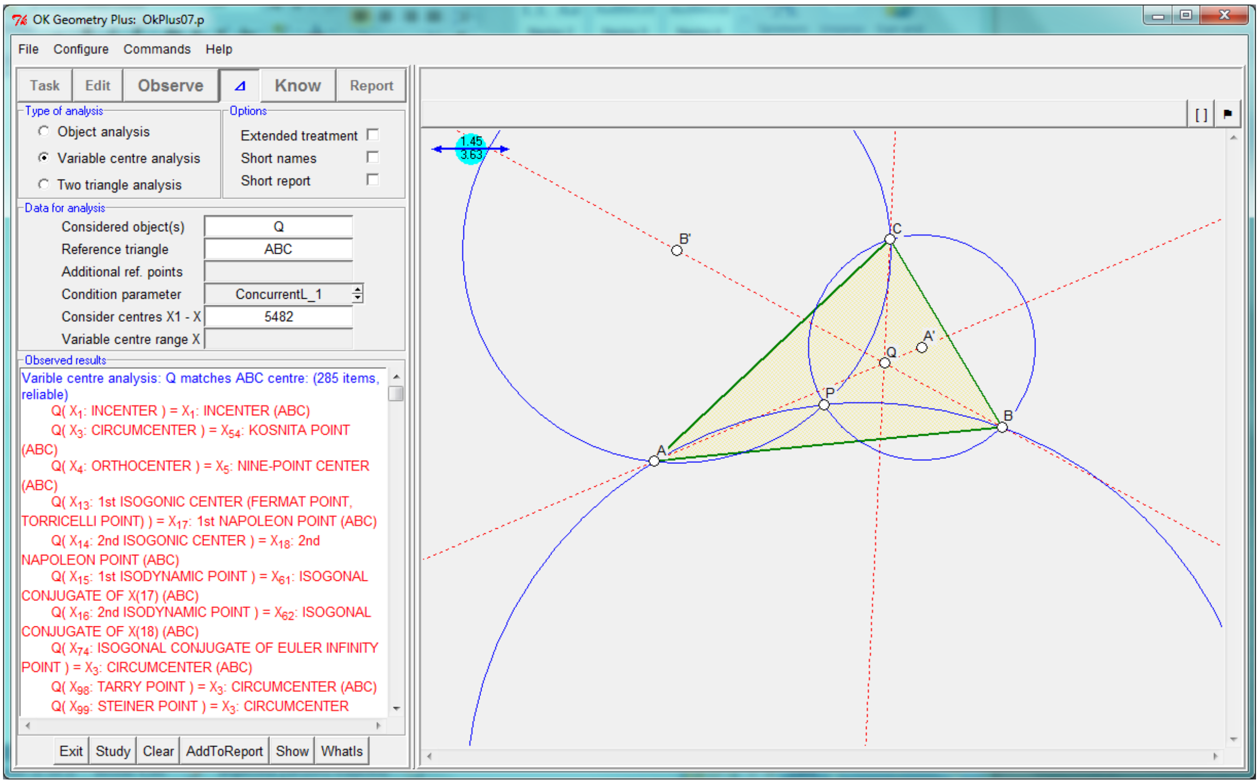


Figure 24