

# OK Geometry Plus

Reference for OK Geometry Plus (v.20)

Zlatan Magajna

June 2024

---

# Contents

<b>1</b>	<b>Introduction.....</b>	<b>4</b>
<b>2</b>	<b>Observing and creating triangle objects .....</b>	<b>5</b>
2.1	<i>The help system .....</i>	6
2.1.1	The Glossary .....	6
2.1.2	Triangle commands .....	8
2.2	<i>Simple observation of triangle objects .....</i>	8
2.3	<i>Special commands for triangles.....</i>	11
2.4	<i>Cyclic constructions.....</i>	19
2.4.1	Interactive detection of cyclic perspectivities.....	22
2.5	<i>Examples.....</i>	25
2.6	<i>Advanced triangle analysis .....</i>	33
2.6.1	Accessing the Triangle analysis module .....	34
2.6.2	Triangle analysis scheme .....	34
2.6.3	Observed results .....	38
2.6.4	Object analysis .....	40
2.6.5	Two triangle centres.....	41
2.6.6	Rule-points and 2-rule-points analysis .....	42
2.6.7	Triangle perspectivity analysis .....	44
2.7	<i>Quadrilateral objects.....</i>	47
2.7.1	Basic concepts .....	47
2.7.2	Special commands for quadrilateral objects.....	49
<b>3</b>	<b>Advanced query of objects.....</b>	<b>52</b>
3.1	<i>Query commands.....</i>	52
3.2	<i>Known, unknown and auxiliary objects .....</i>	53
3.3	<i>Advanced query examples .....</i>	56
<b>4</b>	<b>Observing algebraic relations (formulae) .....</b>	<b>62</b>
4.1	<i>Introduction .....</i>	62
4.2	<i>Observe formulae – simple .....</i>	64
4.2.1	Comments on observed formulae.....	66
4.2.2	Setting the parameters for observing formulae.....	66
4.2.3	Managing the results of Observe formulae.....	67
4.2.4	More examples.....	68

4.3	<i>Observe formulae – advanced</i> .....	70
4.3.1	Comments on observed formulae .....	72
4.3.2	Symbolising geometric quantities .....	73
4.3.3	More examples.....	74
4.4	<i>Observing formulae - triangle</i> .....	75
4.4.1	Comments on observed formulae.....	77
4.4.2	Geometric quantities of the reference triangle .....	78
4.4.3	More examples.....	79
<b>5</b>	<b>Generic constructions</b> .....	<b>81</b>
5.1	<i>Introduction</i> .....	81
5.2	<i>Working with generic constructions (an overview)</i> .....	83
5.2.1	Attaching rules to operations.....	84
5.2.2	Accessing examples of generic constructions .....	85
5.2.3	Creating a selection of examples .....	87
5.2.4	Fast scan of conditions criteria.....	89
5.2.5	Observing properties in examples.....	90
5.3	<i>Rules (generic operations)</i> .....	92
5.3.1	Rules for creating simple objects .....	96
5.3.2	Rules for transforming simple objects .....	98
5.3.3	Triangle A-points .....	99
5.3.4	Rules for triangles and quadrilaterals on a segment.....	101
5.3.5	Rules for triangle objects.....	101
5.3.6	Rules for quadrilateral objects .....	102
5.3.7	Editing rules.....	103
5.4	<i>Examples of generic constructions</i> .....	104
5.5	<i>Triangle analysis of generic constructions</i> .....	108
5.6	<i>Formulae observation in generic constructions</i> .....	112
<b>6</b>	<b>Using the Working archive of constructions</b> .....	<b>116</b>
<b>7</b>	<b>Using the GeoGen archives of constructions</b> .....	<b>117</b>
7.1	<i>Introduction</i> .....	117
7.2	<i>Using GeoGen from OK Geometry</i> .....	119
7.3	<i>Inspecting GeoGen archives</i> .....	123
7.4	<i>A second example</i> .....	124
7.5	<i>Additional comments</i> .....	126

## 1 Introduction

OK Geometry Plus is an extension of OK Geometry Basic. This document should therefore be thus considered a continuation of the OK Geometry Basic references.

OK Geometry Plus consists of several independent modules that are functionally integrated into OK Geometry Basic.

The first module is dedicated to **triangle geometry**. It deals with many objects and operations commonly used in triangle geometry. It contains also elements of geometry of quadrilaterals. Not only can you construct such objects, but you can also observe how a geometric object relates to an extensive set of triangle objects.

The second module provides a more **in-depth query** of objects in a dynamic construction. It relates a given object not only to other objects in the configuration, but also to many other objects derived from the given object. Thus, it can be of great use in hypothesising solutions to construction problems.

The third module **observes algebraic relationships** between geometric quantities in dynamic constructions. It can help you hypothesise how a geometric quantity (distance, radius of a circle, area of a triangle, etc.) is related to other geometric quantities in the configuration.

The fourth module deals with **generic constructions**. You can think of generic constructions as families of constructions obtained with different operations at certain construction steps. This module allows you to create families of dozens or even hundreds of different constructions at the same time. The entire family of constructions can be observed simultaneously, you can also perform triangle analysis or algebraic observation on the family.

Finally, we mention the exciting possibility of connecting OK Geometry with the **GeoGen** program, developed by **Patrik Bak**, which enables the creation of demanding proving problems in planar geometry. The **GeoGen** program can be called from OK Geometry, the generated configurations and proving tasks can then be reviewed and analysed in OK Geometry.

The Plus mode of OK Geometry can be accessed via the command *Configure/Working mode/Plus* in the main menu.

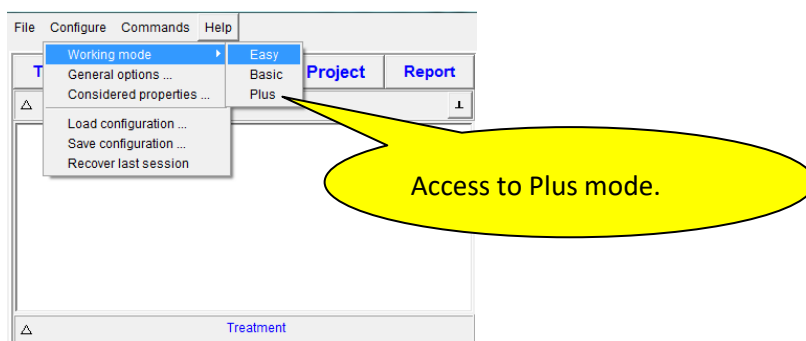


Figure 1

## 2 Observing and creating triangle objects

The core of OK Geometry Plus is a large database of

- triangle centres (e.g. symmedian point, currently about 60 000 of them),
- triangle related triangles (e.g. orthic triangle, currently more than 230 of them),
- triangle related lines (e.g. Euler line, currently 25 lines, more that 2500 including those of related triangles),
- triangle related circles (e.g. 9-point circle, currently 92 circles, more than 7500 including the characteristic circles of related triangles),
- triangle related conics (e.g. Feuerbach hyperbola of a triangle, currently 39 conics),
- triangle cubics (e.g., Neuberg cubic of a triangle, currently more than 1000 of them),
- triangle operations that associate a point, line, circle, triangle, or conic to a given point (e.g. isogonal conjugation of a point, currently 49 operations),
- quadrilateral objects (e.g., Euler-Poncelet point, Newton line, currently about 100 objects).

The database elements can be used directly in constructions as well as in observation of objects. The Plus mode comprises three main procedures:

- Simple observation of triangle objects
- Using triangle objects in constructions, cyclic constructions
- Advanced observation of triangle objects

In the final section we present some examples.

Triangle geometry involves a multitude of concepts (various objects and operations). In order to facilitate the work, a help system was developed. It is explained in the Section 2.1.

Triangle objects refer to a specific triangle under consideration. To simplify the work, one can declare a **reference triangle** to which triangle objects refer to by default (see Section 2.3). When possible, it is a good practice to set the reference triangle at the very beginning of a construction. You can choose between two common ways of labelling the vertices of the reference triangle (Figure 2). At the same time you can also choose the way of labelling triangle centres, e.g.,  $X(5)$ ,  $X_5$ , or simply 5. Both can be set with *Configure/General options/Sketch/Triangle centre notation* command.

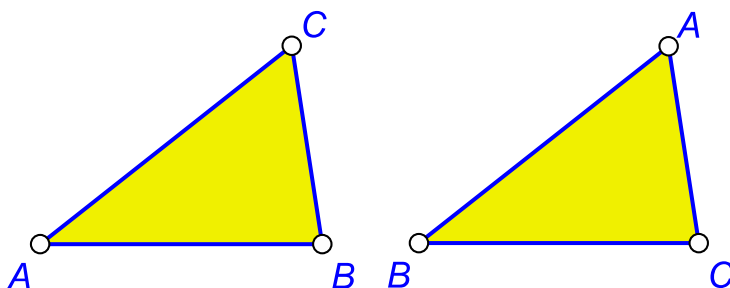


Figure 2

## 2.1 The help system

The help system of OK Geometry Plus consists of two tools: *Triangle commands* and (triangle) *Glossary*. Both tools are accessible in all triangle related texts in OK Geometry, including in texts inside the tools themselves.

### 2.1.1 The Glossary

The (triangle) Glossary can be accessed with the command *Help/Glossary* or with the **F1** key. For a given term the glossary provides a list of concepts related to that term. We explain how to use of glossary with an example.

Suppose we come across the term ‘Gergonne point’. We may not know what it is and where to look in OK Geometry for the related commands. We call up the Glossary (by pressing **F1**) and enter the (approximate) term - for example ‘gergone’.

We now explain the form that appears (see Figure 3).

All entries related to the term under consideration are grouped into three categories (see the red arrow in Figure 3):

**Commands.** Here are grouped the standard commands of dynamic geometry that relate to the term under consideration (e.g. Perpendicular segment, Arc 3pts). Clicking on an entry displays the help for the command. With a right-click on the description you can also access the command. In our case, the group of commands referring to ‘gergone’ (or something similar) is empty.

**Special.** Here are grouped the entries from **triangle geometry** that refer to the term under consideration. In most cases the entries correspond to a command from the Sketch Editor menu Special, but some entries are only explanation of a term. The group includes also some basic **quadrilateral objects** (see section 2.7) from the *Encyclopedia of quadric-figures*<sup>1</sup>, their names begin with QA-, QL or QG-. In our case, 12 entries were found. In the first 4 of them, ‘gergone’ (or something similar) is included in the name of the entry/command. The remaining entries have a ‘-’ prefix since ‘gergone’ (or something similar) is included only in the explanatory text of the entry. With a right-click on the description you can access each command.

**ETC.** Here are grouped the centres from the *Encyclopedia of triangle centers*<sup>2</sup> that contain the string ‘gergone’ (or something similar) in their name. There is no explanatory text for these entries, but you can access the commands for constructions these centres via the glossary. In our case this group consists of about 35 entries.

**CTC.** Here are grouped those triangle cubics from the *Catalogue of triangle cubics*<sup>3</sup> that contain the string ‘gergone’ (or something similar) in their name. In most cases there is no explanatory text for these entries, but you can access the commands for constructions these cubics via the glossary. In our case this group consists of 3 entries.

---

<sup>1</sup> <https://chrisvantienhoven.nl/mathematics/encyclopedia>

<sup>2</sup> <https://faculty.evansville.edu/ck6/encyclopedia/etc.html>

<sup>3</sup> <http://bernard-gibert.fr>

So, entries of the glossary for triangle geometry can be found under the category **Special**. Clicking any article (entry) will take you to the information in that article. In our case, we click on the 2<sup>nd</sup> entry ('Gergonne point'), and the form is filled in, see Figure 3.

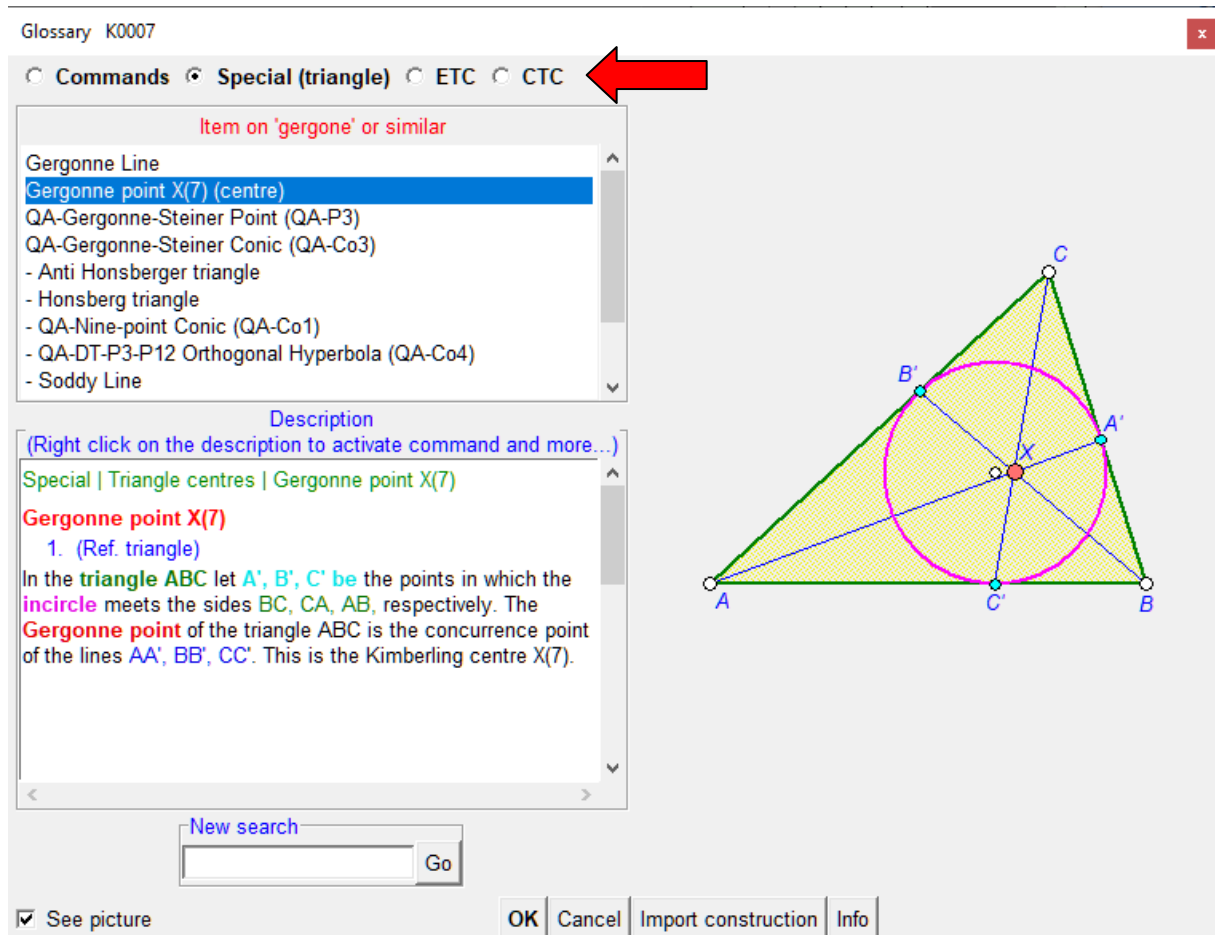


Figure 3

The form contains a **description** and an **illustration** of the selected term. Since there is a command related to the Gergonne point, the description also contains the location of the command in the menu (first line) and the arguments of the command.

**A right click on the description** leads to a menu with further help.

**Explain terms** turns the cursor into a **question mark with an arrow**. A click of the arrow on a term in the description (e.g. 'incircle') shows an explanation of the meaning of the term (possibly including the adjacent words).

**Glossary help** turns the cursor into a **spider**. A click of the spider on a term in the description activates the glossary for that term.

**Execute command** activates the considered command (some commands can be executed only in the Sketch Editor). In our case the list of commands for triangle centres appears with the command *Gergonne point* selected.

### 2.1.2 Triangle commands

The *Help/Triangle commands* displays the list of all commands for the creation of triangle related objects and operations. The list can be restricted to commands that create a specific type of objects. In the Filter entry write the exact word to look for (e.g., 'gergonne') or an approximate one with a question mark at the end (e.g., 'gergone?'). For each command there is a description, which has the same functionality as the description in the glossary. The command can be therefore activated by a right click on the description.

## 2.2 Simple observation of triangle objects

We describe here how to relate constructed objects to a specific triangle – called the **reference triangle**.

We explain this with a simple example<sup>4</sup>.

### Example

Let  $A'$ ,  $B'$ ,  $C'$  be the base points of the altitudes from the vertices  $A$ ,  $B$ ,  $C$  of the triangle  $\triangle ABC$ . Furthermore, let  $A''$ ,  $B''$ ,  $C''$  be the midpoints of  $B'C'$ ,  $C'A'$ , and  $A'B'$ . By observing the configuration or otherwise, we note that the lines  $AA''$ ,  $BB''$ ,  $CC''$  (apparently) meet at a common point  $P$ . We wonder how is this point related to  $\triangle ABC$  (

Figure 4).

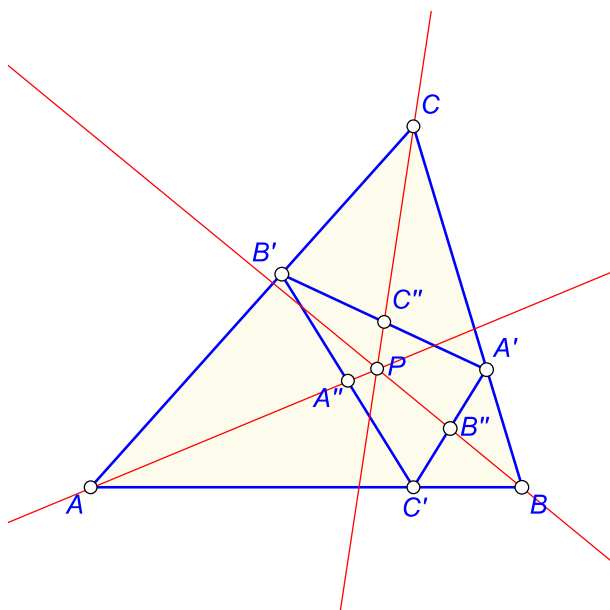



Figure 4

If we are not in Plus mode, we need to switch to it – from the menu at the top select the command *Configure/Working mode/Plus*.

Then click on the command  (*Analyse objects in reference triangle*) on the taskbar on the right.

<sup>4</sup> OKExamples\OKG\_Plus\Triangle\_01.p



If the reference triangle has not been defined yet, first enter the labels of the points that define the reference triangle (Figure 5). The reference triangle is coloured pale yellow. You can declare the reference triangle at any time with the command *Comands/Set reference triangle*.

Then click on the objects you want to know something about, in our case the point P.

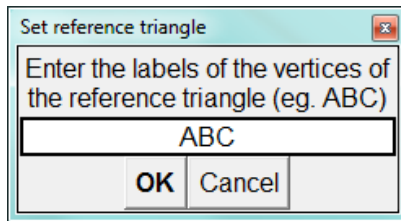


Figure 5

After a short calculation a form appears with the observational results (Figure 6).

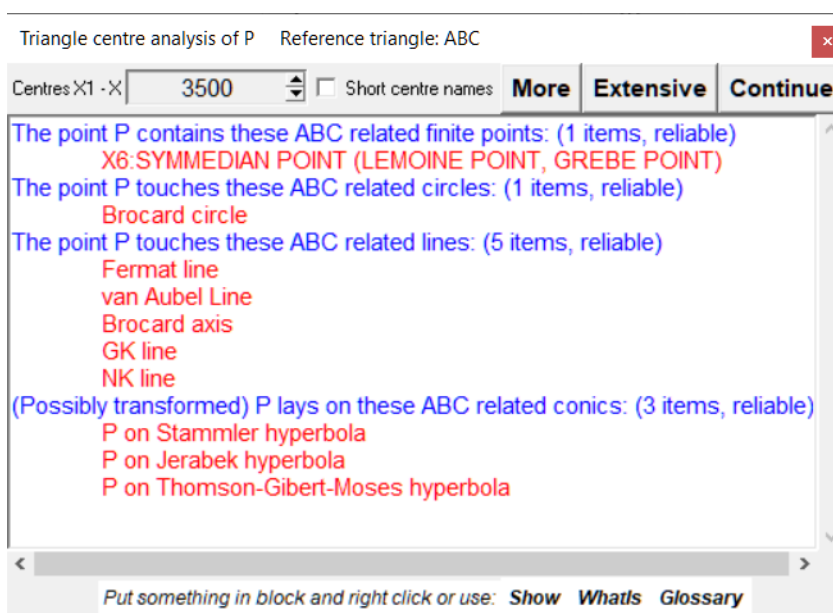


Figure 6

OK Geometry observes that the point P appears to be the symmedian point (commonly denoted as X6) of the triangle  $\triangle ABC$  and presents some additional observed facts related to the point P. To view more information about the object P (more properties and more considered objects), click on the **More** button. The **Extensive** button will display an extensive list of properties involving a large set of triangle objects.

In the form (Figure 6) we can specify the range of centres (from the ETC list of triangle centres<sup>5</sup>) to be used in the analysis. For the most exhaustive analysis choose the option 16342+bic, which takes into account all the first 16341 centres, about 200 bicentres and many centres (up to the index 60000). The option \*59992 takes into account the first 59992 centres, but performs only a limited and less reliable analysis.

<sup>5</sup> ETC is acronym for Encyclopedia of Triangle Centers  
(<https://faculty.evansville.edu/ck6/encyclopedia/ETC.html>).

The objects that are mentioned in the list of properties can be readily displayed. For example, to display the Fermat line, put the text Fermat line in a block (or simply position the cursor on the text) and click the **Show** command – the Fermat line will appear on the construction (Figure 7). To add at once several objects mentioned the list to the construction, simply put the objects in a block and click the **Show** command.

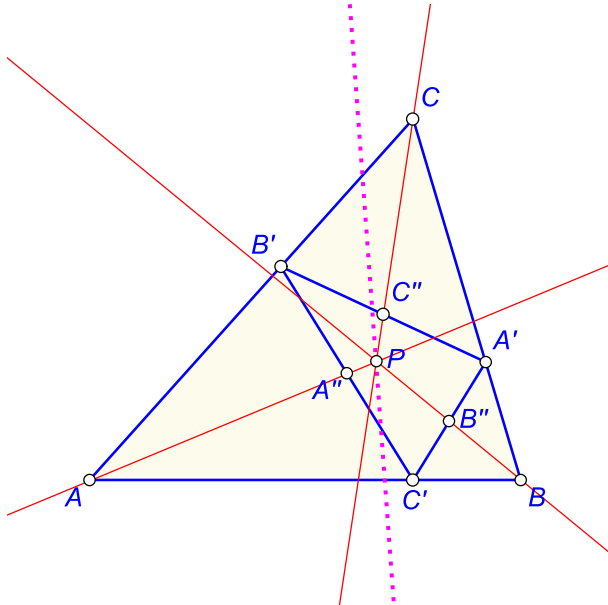


Figure 7

You can also obtain an explanation of the terms in the list of properties. For example, to obtain some basic information of the Fermat line (or some other term), click on the **WhatIs** command (a question mark with arrow will appear) and then click on the term with the (question mark) pointer. An explanation of the Fermat line will appear (Figure 8).

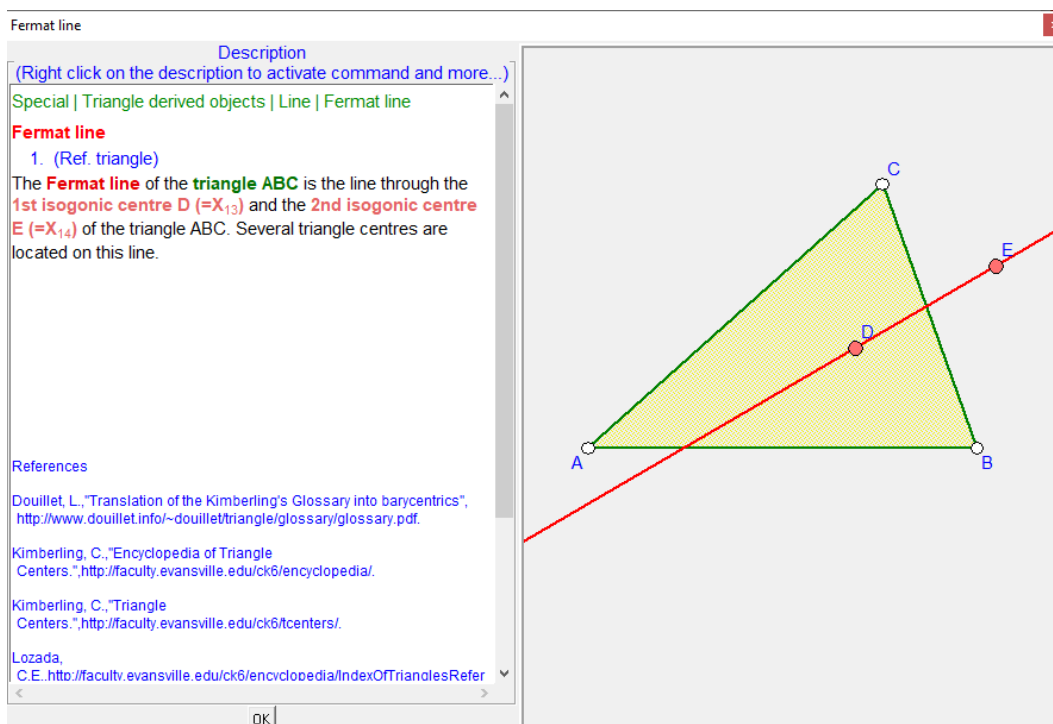


Figure 8

Finally, there is a glossary that relates the terms in the list of properties to other similar terms in triangle analysis. For example, to find articles related to symmedian point, click on the **Glossary** command (a spider pointer will appear) and then click on the term Symmedian point with the (spider) pointer. The list of articles related to symmedian will appear (Figure 9).

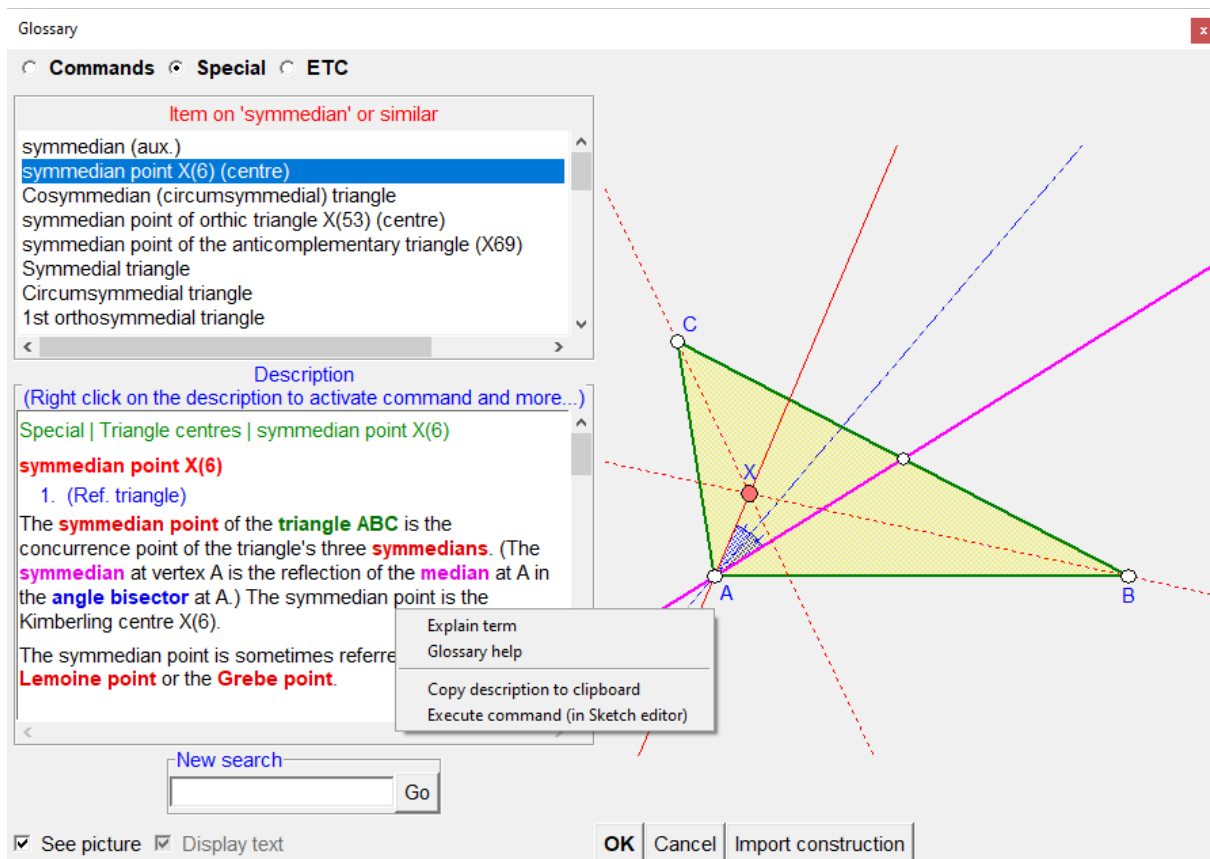


Figure 9

Some additional commands related to the form (Figure 6) are available with a right-click on the Description field.

## 2.3 Special commands for triangles

The Sketch Editor in Plus mode has an additional group of commands named **Special**. The vast majority of commands in this group are specific to triangle geometry and most of them have similar syntax.

The Special commands allow you to construct various objects related to a triangle. The triangle the objects refer to is called the **reference triangle**. Since the same triangle is often used repeatedly as a reference triangle, it is common to declare a **default reference triangle** and use it for most commands in the group. This is done with the command *Commands/Set reference triangle* or *Special/Set reference triangle*.

Figure 10 shows a typical group of commands (characteristic objects of a triangle).

- In the first line you select the type of object you want to create (a line, in our case).

- The commands in this group (with a few exceptions) refer to the default reference triangle, but it is always possible to specify a different reference triangle *as the first argument of the command*. Note the red line with the checkmark at the bottom, which indicates that the default reference triangle will be used unless the checkmark is removed.
- Also note the checkmark *See picture*. When this checkmark is set, a pictorial representation of the object appears on the right side.

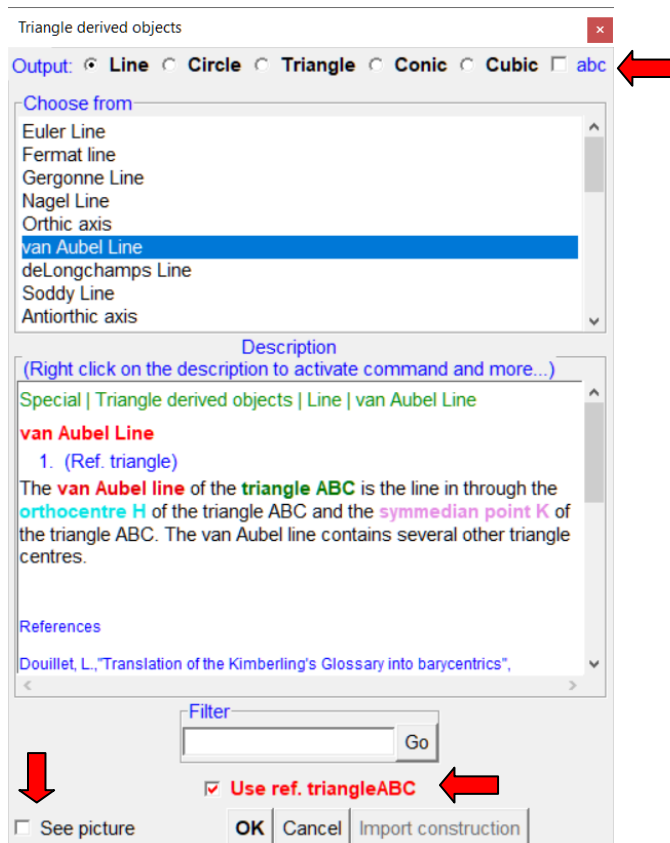


Figure 10


In many cases the form contains dozens, hundreds, or even more (sub)commands. Sometimes they are grouped by the type of object to be created. To make the selection easier, you can **arrange** the (sub)commands alphabetically. There is also a **filter** to reduce the number of entries in the list. If you enter 'malfatti' in the filter, only entries that contain the string 'malfatti' will be displayed. If you type 'malfati?', only entries containing something similar to 'malfati' will be displayed in the list.

In the Special menu the commands are grouped according to the input objects:

<p><i>Triangle centres</i></p>	<p>Triangle centres are organised in five lists:</p> <ul style="list-style-type: none"> <li>- <b>simple centres</b> - the four classic triangle centres;</li> <li>- <b>advanced centres</b> - the most known centres, e.g. isodynamic point;</li> <li>- <b>ETC centres</b> - the first 16341 centres of the ETC list of triangle centres;</li> <li>- <b>ETC+ centres</b> - the centres from 16342 on of the ETC list of triangle centres;</li> <li>- <b>bicentric</b> - some well known triangle bicentres, e.g., the Brocard points.</li> </ul> <p>A standard notation is used for the centres from the ETC list: <math>X_n</math>, <math>Xn</math> or <math>X(n)</math> for the <math>n</math>-th centre.</p>
<p><i>Triangle derived objects</i></p>	<p>The classic triangle objects, other than points, are organised in several lists:</p> <ul style="list-style-type: none"> <li>- <b>triangle lines</b>, e.g., the Euler line;</li> <li>- <b>triangle circles</b>, e.g., the Brocard circle;</li> <li>- <b>triangle triangles</b>, e.g., the orthic triangle;</li> <li>- <b>triangle conics</b>, e.g., the Kiepert hyperbola;</li> <li>- <b>triangle cubics</b>, e.g., the Neuberg cubic.</li> </ul>
<p><i>Triangle/Point derived objects</i></p>	<p>Various objects can be associated to a point with respect to a reference triangle. Such objects are organised in several lists, depending on the type of the objects. The considered type of resulting objects are:</p> <ul style="list-style-type: none"> <li>- <b>points</b>, e.g., the isogonal conjugation of a point in a reference triangle;</li> <li>- <b>lines</b>, e.g., the trilinear polar of a point in a reference triangle;</li> <li>- <b>circles</b>, e.g., the Cevian circle of a point;</li> <li>- <b>triangles</b>, e.g., the pedal triangle of a point in a reference triangle;</li> <li>- <b>conics</b>, e.g., the inconic of a point.</li> </ul>
<p><i>Triangle/Point/Point derived objects</i></p>	<p>The objects that are defined by 2 points with respect to the reference triangle are organised into these groups:</p> <ul style="list-style-type: none"> <li>-points, e.g., cross conjugate of two points;</li> <li>-conics, e.g., bicevian conic of two points.</li> </ul>

<p><i>Objects by triangle centres</i></p>	<p>This command allows an effective way to define and visualise triangle related objects passing through specified triangle centres. Simply enter the centres as indicated by the following examples:</p> <p>5            ETC centre X(5), nine points centre</p> <p>1-100       The first 100 centres (non all labelled)</p> <p>(2,14)      The line through X(2) and X(14)</p> <p>(2,14,15)   The circle through X(2),X(14),X(15)</p> <p>(5,6,7,8,9) The conic through X(5),X(6),X(7),X(8),X(9)</p> <p>(1,3,4,5,17,18)   The cubic through the vertices of the reference triangle and X(1),X(3),X(4),X(5),X(17),X(18)</p> <p>2,4,(3,6)    The centres X(2) and X(4) and the line through X(3) and X(6)</p>
<p><i>General triangle derived point</i></p>	<p>With this command you can analytically define points, point transformations or point operations in a reference triangle. In the definition you can use barycentric, trilinear or tripolar coordinates as well as some triangle parameters.</p> <p>See Figure 12 for an example.</p>
<p><i>Various constructions</i></p>	<p>Here are collected various constructions of points, lines, circles, conics, and cubics. The commands are grouped into categories (see the first line in the form) according to the type of constructed object. The arguments of the commands vary from command to command. However, if the first argument of a command is a triangle then you can use the reference triangle as the first argument (by checkmarking the appropriate field, see Figure 10).</p> <p>In some commands you can add to the construction also auxiliary objects.</p> <p>Here are just some examples of commands:</p> <ul style="list-style-type: none"> <li>- <b>perspectivity centre</b> of two triangles (if it exists);</li> <li>- <b>insimilicentre</b> of two circles;</li> <li>- <b>polar</b> of a point with respect to a conic;</li> <li>- <b>eulerologic centre</b> of two triangles (if it exists);</li> <li>- <b>perspeconic</b> conic of two perspective triangle.</li> </ul>

<p><i>Set reference triangle</i></p>	<p>With this command you define the reference triangle, which is (optionally) used by default in other special commands. The vertices of the reference triangle must be labelled.</p> <p>To clear the reference triangle enter in the form an empty line.</p> <p>OK Geometry uses standard notation for geometric quantities that refer to the reference triangle (or to the triangle that takes the place of the reference triangle). If the vertices of this triangle are PQR (in that order), then <math>a</math>, <math>b</math>, <math>c</math> denote the lengths of the sides <math>a =  QR </math>, <math>b =  RP </math>, <math>c =  PQ </math>. The sizes of the angles are denoted by A (size of the angle at P), B (size of the angle at Q), C (size of the angle at R) and so on.</p>
<p><i>A triangle</i></p>	<p>This command is active only at the very beginning of the construction. It generates a 'random' triangle ABC with labelled vertices and its cyclic structure. You can set the way of labelling the vertices in <i>Configure/General options/Sketch/Triangle centre notation</i>.</p>
<p><i>Declare cyclic objects</i></p>	<p>With this command you declare three objects (of the same type) as cyclic in accordance to the current cyclic construction of the reference triangle. When the Sketch Editor is creating a 'cyclic construction', any command that involves one or more cyclic objects is repeated cyclically on the related triads of objects.</p> <p>Cyclic structure often significantly simplifies geometric constructions on triangles. Please, refer to Section 2.4 for more.</p>
<p><i>Cyclic construction</i></p>	<p>Turns the <b>Cyclic</b> construction flag ON and OFF. At ON, the display pointer takes the shape similar to a small circle and the commands are executed cyclically with respect to the reference triangle.</p> <p>Cyclic structure often significantly simplifies geometric constructions on triangles. Please, refer to Section 2.4 for more.</p>
<p><i>Detect cyclic perspectivities</i></p>	<p>Turns ON/OFF automatic detection of cyclic perspectivities. When this flag is ON, whenever in a cyclic construction a point (actually, a triplet of points) is constructed it is automatically checked whether the triangle from the constructed triplet of points is in some way perspective to the reference triangle. Please, refer to Section 2.4.1 for details.</p>

<p><i>Analyse object wrt. triangle</i></p>	<p>The command (available also as  button in tool-bar) executes a simple observation of the object you click on (see Section 2.2). If no reference triangle has been defined yet, you will be first asked to specify the reference triangle. Then click on an object (existing point, line, circle, conic, cubic). After a while, OK Geometry displays a form with a list of observed properties of the object in relation to the reference triangle. Click on the More or Extensive button for a longer or an extensive list of observations. To inspect other objects, click <b>Continue</b> and select the next object to be analysed. Here are some illustrative observations:</p> <ul style="list-style-type: none"> <li>- The selected point is the centre X(42) of the reference triangle.</li> <li>- The selected line is tangent to the 9-point circle of the reference triangle.</li> <li>- The selected point is the intersection of the lines X(6)X(9) and X(5)X(7).</li> <li>- The selected line passes through X(21) and the isogonal conjugate of X(35).</li> </ul> <p><b>Note.</b> See Section 2.2 for an exhaustive description of the command.</p> <p><b>Note.</b> Right click on the displayed results: there are several available commands to visualise and obtain information about the listed objects (place the cursor set a block in the text appropriately).</p> <p><b>Note.</b> This command is intended for a quick analysis. For a more elaborated analysis, see Section 2.6.</p>
<p><i>Observe formulae ...</i></p>	<p>The command starts the form for the observation of algebraic relations between geometric quantities in the reference triangle.</p> <p>Please, refer to Section 4.4 for more.</p>

All commands for triangle centres and other objects related to a triangle are used in a similar way. Suppose we are looking for the circumcevian triangle of a given point with respect to the reference triangle. Since our object is defined by the reference triangle and a point, choose the **Triangle/Point derived objects** command. In the form that appears (Figure 11), first select the 'Triangle' option (since the resulting object is a triangle). From the listed operations, select the option Circumcevian. **Notice the option in the lower left corner: If a reference triangle was previously defined, specify whether you want to use the already defined reference triangle or another triangle.** As you can see, the form contains a short description of the operation with an optional illustration (option 'See picture').



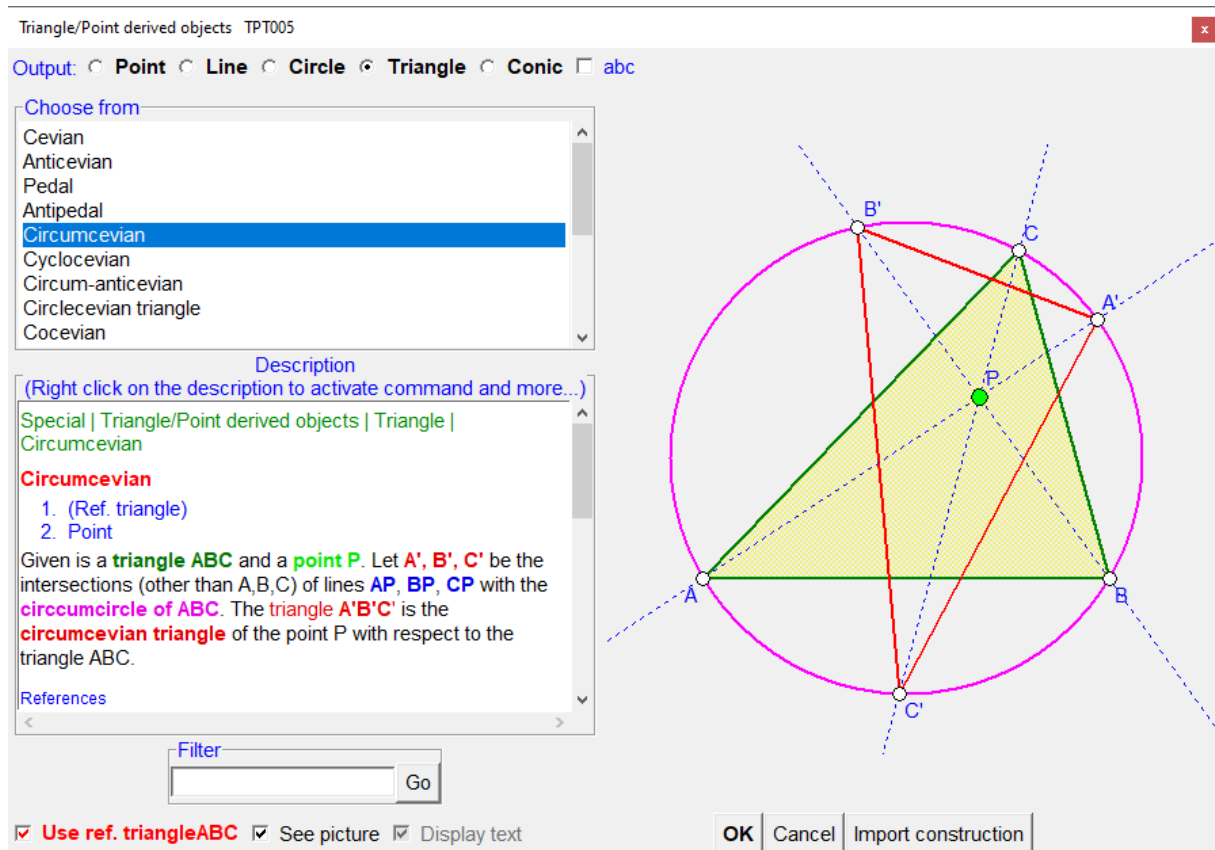


Figure 11

Here is a detailed description of the steps to follow:

1. Select the type of the resulting object (in the above case choose between Point, Line, and Triangle). A list of available objects/operations appears. Select '**abc**' option if you want the list to be arranged alphabetically.
2. Select the object or operation from the displayed list. A short description of the object/operation appears. If you want an illustration of the description, select the '**See picture**' option.
3. If a reference triangle has previously been defined, you can use it as a reference triangle for the operation. (You don't see this check box if the reference triangle has not been set.)
4. Click OK and select the involved objects:

If 'Use ref. triangle' is <b>ON</b>	If 'Use ref. triangle' is <b>OFF</b> or is not present
<ol style="list-style-type: none"> <li>1. Since the reference triangle already defined, you only need to select the additional arguments (depending on the command, there can be none, one point or two points).</li> </ol>	<ol style="list-style-type: none"> <li>1. ALWAYS specify the reference triangle first. Either pick its three vertices OR pick the polyline that defines the triangle.</li> <li>2. Select the additional arguments (depending on the command, it can be none, one point or two points).</li> </ol>

The usual way to obtain a triangle object is as follows:

- Consider the defining objects of the object to be created: this can be a triangle, a triangle and a point, a triangle and 2 points. Then choose the command *Triangle Centres* or *Triangle Derived objects*, *Triangle/Point objects*, *Triangle/Point/Point objects*.
- In the form select the type of objects you want to create (Point, Line, Circle, Conic, Cubic).
- Select the triangle object from the list of objects.
- Checkmark whether you want to use the existing reference triangle or not.
- Click the button OK.
- If you decided not to use the reference triangle, set the triangle as the first argument, either by clicking its vertices or by clicking a corresponding polyline.
- Click the remaining point type arguments if necessary.

Some constructions, that are sometimes used in triangle geometry and do not comply with the above description, are grouped in the *Various constructions* command.

To get to the desired command you can always use the *Help/Triangle commands* help or the **F1** button on the keyboard (glossary).

The basic objects in triangle geometry are the triangle centres. You can get a triangle centre in several ways:

- via the triangle name (e.g., Nagel point or X8) using the command *Triangle Centres*;
- via the ETC number of the centre with the command *Objects by triangle centres* (you can draw several centres and other triangle objects at once);
- via the trilinear, barycentric or tripolar coordinate of the point with the command *General triangle derived point*.

In the following example (Figure 12) we define analytically a point relative to the reference triangle. Let the barycentric coordinates of the point be  $(b^2+c^2, c^2+a^2, a^2+b^2)$ , where  $a, b, c$  are the sides of the reference triangle.

General triangle centre / transformation

Type of command

☒ Triangle centre ☐ Triangle-Point transformation ☐ Triangle-Point-Point transformation

Used triangle coordinate system

☒ Barycentric ☐ Trilinear ☐ Tripolar1 ☐ Tripolar2

Output point coordinates

Coordinate 1

Coordinate 2

Coordinate 3

Explanation

In the expressions use the following notation:

a,b,c ..... triangle sides  
A,B,C or alpha, beta, gamma ..... triangle angles  
omega or W ..... Brocard angle  
r, R ..... radius of inscribed and circumscribed circle  
area ..... area of the triangle  
S, SW, SA, SB, SC ... Conway notation  
..... (2\*area, S/tan(omega), S/tan(A), etc.)  
OI, OH ..... distance between O=X3, I=X1, H=X4  
s, sa, sb, sc ... half-perimeter. s-a, s-b, s-c

☒ Use ref. triangleABC

Figure 12

If you want to define a point relative to another triangle (and not to the current reference triangle), uncheck the 'Use of reference triangle' box. After clicking on the OK button you will be asked to specify the vertices of the vertices of this triangle.

## 2.4 Cyclic constructions

Triangle constructions are often cyclic in the sense that they take into consideration the cyclic structure of the triangle. The cyclic structure consists of ordered triads of objects of a considered triangle. We say that an operation is performed cyclically if the involved arguments in the repeated operation are permuted cyclically according to the cyclic structure. Thus, a cyclic operation in a construction creates a triad of objects of the same type and the triad is added to the existing cyclic structure. It is good practice to avoid triads with identical objects.

The Sketch Editor command *Special/A triangle* creates a triangle with cyclic structure consisting of three vertices and three sides of the triangle. The **Cycl** flag indicates whether the current operation is to be executed as cyclic or normal (Figure 13). When the **Cycl** flag is ON, it is bright and the mouse cursor takes the shape of a small circle. You can change the **Cycl** flag with a click on it or using the command *Special/Cyclic construction*. When a command is executed as cyclic, the resulting triad of objects is automatically added to the cyclic structure. You can add a triad of object to the cyclic structure also with the command *Special/Declare cyclic objects*.



Figure 13

By considering the cyclic structure in triangle constructions makes the construction process much easier and faster. Also, the resulting figure is easier to understand, because the 'cyclic copies' of the constructed objects are bleached. (You can easily remove the bleaching with the command *Action/Emphasise/Unmark all bleach*.)

The easiest way to create cyclic constructions is to start with the command *Special/A triangle*. Then continue with other commands – but be sure to turn ON the **Cycl** flag ONLY when you want the operation to be cyclic - otherwise you will get identical copies of objects. In most cases, but not allways, OK Geometry corrects such an error (i.e, ignores the cyclic flag) – in such cases the cursor turns from circle form to the square form for a fraction of a second.

Note that some command clear the **Cycl** flag automatically, e.g., the *Polyline* or *Object by triangle centres*, while some other commands ignore the **Cycl** flag, e.g. the commands for labelling or restyling objects.

We illustrate how to do a cyclic construction with a simple example<sup>6</sup> (Figure 14). On each side of the triangle ABC we construct outwardly an equilateral triangle. Let the added vertices be A', B', C'. It is known that the lines AA', BB', CC' concur in a point Q (the 1<sup>st</sup> isogonic point). Moreover, let P be the centre of the circle through the points A', B', C'.

<sup>6</sup> OKExamples\OKG\_Plus\Triangle\_02.p

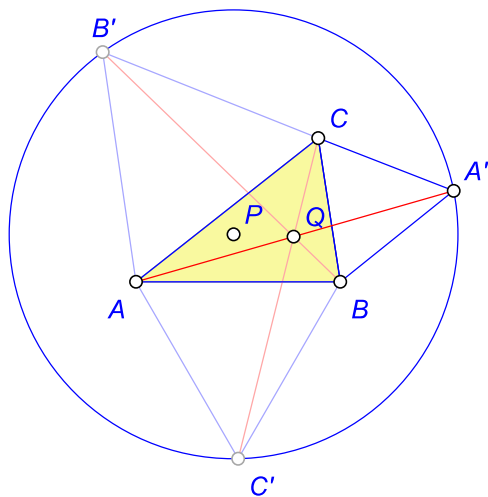


Figure 14

Here are the steps of this cyclic construction.

Step/Command	Comment
<b>Special/A triangle</b>	Obtain a triangle with the initial cyclic structure. Note that this command is active only if no object has yet been constructed.
<b>Special/Cyclic construction</b>	<b>Cyclic construction is turned ON.</b> (Note that the pointer assumes a circular form.) Alternatively, click on the <b>Cycl</b> flag.
<b>Advanced/Shape/Equilateral triangle</b>	Construct the equilateral triangle on the side CB (click vertex C, then B). Simultaneously, bleached equilateral triangles are constructed on the other two sides of the triangle ABC.
<b>Actions/Label vertex</b>	Label the new vertices A', B', C' Note. Labelling vertices is never considered a cyclic operation.
<b>Line/Segment</b>	Construct the segment AA'. Simultaneously two other cyclically defined segments (BB' and CC') are constructed. Note that the three segments concur in a point (1 <sup>st</sup> isogonic centre P, to be constructed).

<b>Special/Cyclic construction</b>	<b>Cyclic construction is turned OFF.</b> This is necessary since command that follows should not be executed cyclically (creating one command per cycle) <sup>7</sup> .
<b>Point/Intersection</b>	Construct the intersection point Q of segments BB' and CC'.  It was necessary to set <b>Cycl</b> OFF, otherwise three coincident intersection points would be constructed (See footnote 7).
<b>Circle/Circle 3 obj</b>	Construct the circle through A', B', C'. Note that <b>Cycl</b> is OFF.
<b>Point/Centre of circle</b>	Construct the point P. (The observation of triangle object P finds that P is X627 = ANTICOMPLEMENT OF X17.) Note that <b>Cycl</b> is OFF. (See footnote 7).

### Example

Here is another nice example<sup>8</sup>: the Dao 6 point circle. Given is a triangle ABC with its centroid G. Let A' be the centre of the circle through B that is tangent to the line AG at G. Let A'' be the centre of the circle through C that is tangent to the line AG at G. Obtain the circle centres B', B'', C', C'' cyclically. The 6 points A', A'', B', B'', C', C'' lay on a circle, called the Dao 6-point circle (Figure 13).

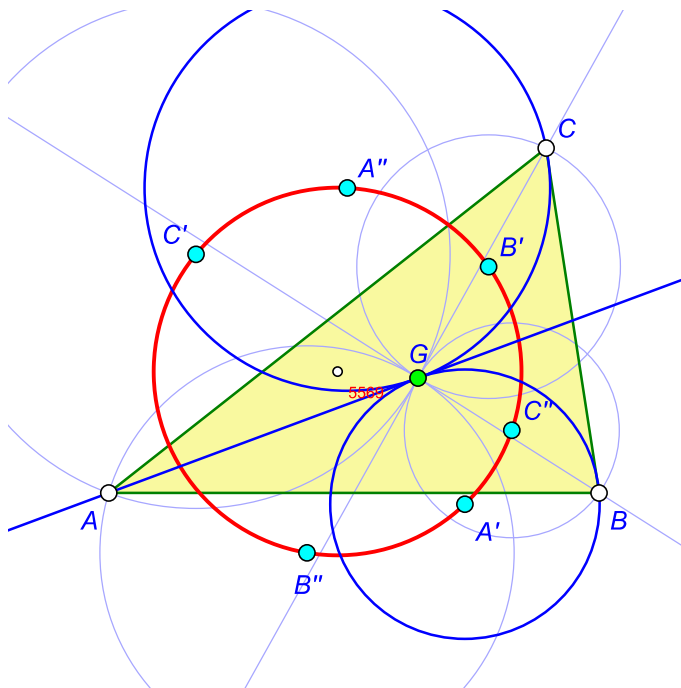


Figure 15

<sup>7</sup> Actually, in this case the system would detect that there is no reason to consider the command cyclically and would ignore the **Cycl** flag.

<sup>8</sup> OKExamples\OKG\_Plus\Triangle\_03.p

Step/Command	Comment
<b>Special/A triangle</b>	Obtain a triangle ABC with the initial cyclic structure. Note that this command is active only if no object has yet been constructed.
<b>Specia/Triangle centres/Advanced/Centroid X(2)</b>	We do this step with the <b>Cycl</b> flag OFF. Label the centroid as G.
<b>Special/Cyclic construction</b>	<b>Cyclic construction is turned ON.</b> (Note that the pointer assumes a circular form.) Alternatively, click on the <b>Cycl</b> flag.
<b>Line/Line 2 points</b>	Pick the points A and G. The lines BG and CG are created automatically.
<b>Circle/Circle 3 objects Point/Centre of circle</b>	Pick the point G, the line AG and the point B. Pick the created circle. At the same time you got 3 circles and then their centres. Label the centres as $A'$ , $B'$ , $C'$ .
<b>Circle/Circle 3 objects Point/Centre of circle</b>	Pick the point G, the line AG and the point C. Pick the created circle. At the same time you got 3 circles and then their centres. Label them as $A''$ , $B''$ , $C''$ .
<b>Special/Cyclic construction</b>	<b>Cyclic construction is turned OFF.</b> We clear the <b>Cycl</b> flag before the Dao circle is drawn (otherwise we would get 3 copies of the circle). Alternatively, click on the <b>Cycl</b> flag.
<b>Circle/Circle 3 objects</b>	Construct the circle through $A'$ , $B'$ , $C'$ .

#### 2.4.1 Interactive detection of cyclic perspectivities

When we study cyclic constructions on a reference triangle, we are often interested in triangles that are in some way perspective to the reference triangle. For this purpose, it is possible to activate a mechanism in OK Geometry that constantly checks whether a triplet of cyclically constructed points

forms a triangle that is perspective to the reference triangle. You activate this mechanism with the command ***Special|Detect cyclic perspectivities***.

Perspectivities of generated cyclic triplets of points with regard to the reference triangle are automatically detected if:

- the reference triangle and other objects in the current construction have a corresponding cyclic structure,
- the **Cycl** flag is activated (***Special|Cyclic construction***),
- the detection of perspectivities is activated (***Special|Detect cyclic perspectivities***).

The cyclic construction of a point now creates 3 cyclically related points, and the software immediately checks whether the resulting triangle is perspective, orthologically perspective, parallelogically perspective, and cyclogically perspective to the reference triangle. The centres of perspectivities are also displayed. The displayed centres of a perspectivity can be inspected as usual. A group of detected centres of perspectivities behave as a single object: if one of the displayed centres is coloured, hidden, deleted, all others displayed centres are coloured, hidden, deleted. If, for example, we only want to work further with one of the centres, we must label it (or put a point on it) and hide (not delete!) the other centres.

**Note.** The interactive method described only detects perspectivities of the reference triangle and triangles with cyclically constructed vertices. To detect perspectivities of any triangle with any of the numerous triangles related to the reference triangle, see Sections 2.6.7.

Here is a simple example (Figure 16). Let  $N$  be the 9-point centre of triangle  $ABC$ . Let  $A'$  be the circumcentre the triangle  $BCN$ . Define  $B'$  and  $C'$  cyclicly.

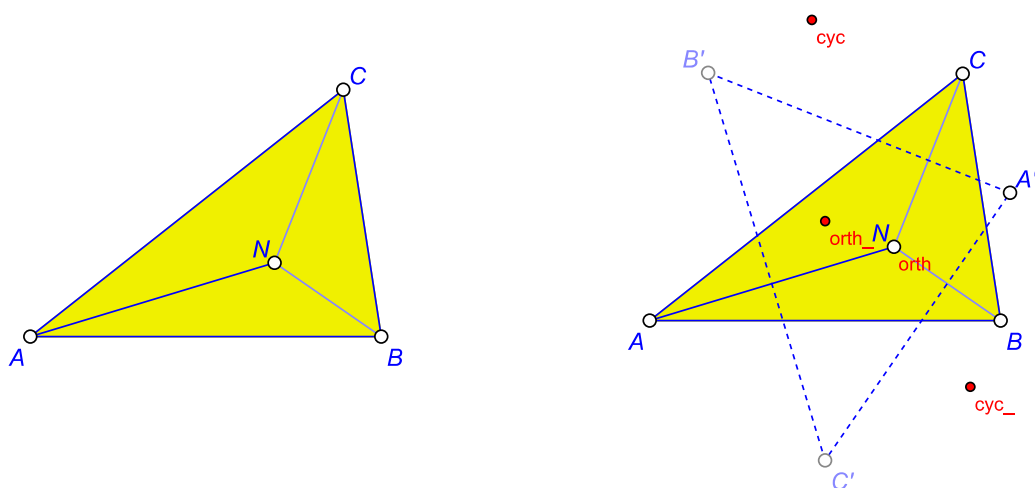


Figure 16

As soon as the point  $A'$  and its cyclic counterparts  $B'$ ,  $C'$  are constructed we are informed that the triangle  $A'B'C'$  is cyclically and orthologically perspective to  $ABC$ . The respective centres of perspectivities are constructed and displayed. The red points 'cyc' and 'cyc\_', for example, are the

centres of cyclogical perspectivity of triangles ABC and A'B'C', thus the circumcircles of triangles AB'C', A'BC', A'B'C meet in the point 'cyc' and the circumcircles of triangles A'BC, AB'C, ABC' meet in the point 'cyc\_'.

Here are the steps of this cyclic construction.

Step/Command	Comment
<b>Special/A triangle</b>	Obtain a triangle ABC with the initial cyclic structure. <b>Note.</b> This command is only active if no object has been constructed yet.
<b>Special/Triangle centres/nine-point centre X(5)</b>	Construct the nine-point centre N of ABC. <b>Note.</b> Make sure to checkmark the entry <i>Use the ref. triangle ABC</i> . The generated point can be immediately labelled by 'N' on the keyboard.
<b>Special/Cyclic construction</b>	<b>Cyclic construction is switched ON.</b> (Note that the pointer takes a circular shape) Alternatively, click on the <b>Cycl</b> flag.
<b>Special/Detect cyclic perspectivities</b>	Switch ON the detection of perspectivities.
<b>Special/Triangle centres/circumcentre X(3)</b>	<b>Note.</b> Make sure to <b>uncheck</b> the entry <i>Use the ref. triangle ABC</i> . Click on points B,C,N. Simultaneously the circumcentres of triangles BCN, CAN, ABN are displayed. The perspectivity centres 'cyc', 'cyc_', 'orth', 'orth_' are also displayed.
<b>Action/Label vertex</b> <b>Line/Segment</b> <b>Restyle objects</b>	Label properly the vertices A', B', C'. Draw (cyclically) lines AN and A'B'.
<b>Special/Analyse object wrt. triangle</b>	We analyse the detected perspectivity centres. The point 'cyc' and 'cyc_', for example, are observed as X1291 and X24772.
<b>Special/Cyclic construction</b>	Cyclic construction flag is switched OFF.
	We would like to work further with centre 'cyc' as the point P and ignore other displayed centres of perspectivities.
<b>Label vertex</b>	The centre 'cyc' is labelled as P.
<b>Action/Hide</b>	Hide the found perspectivity centres. (Deleting the centres would also delete the point P.)



## 2.5 Examples

### Example 1 - The Privalov conic

It is well known that in a given triangle  $ABC$  the vertices of the cevian triangles of any two different points  $D, E$  (not laying on the sidelines of  $ABC$ ) are conconical (Figure 17, left)<sup>9</sup>. We consider the case of two particular points: the Gergonne point ( $D$ ) and the Nagel point ( $E$ ) of a given triangle  $ABC$ . The Gergonne point of  $ABC$  is the point of concurrence of the line segments connecting the points where the incircle of  $ABC$  touches the sides with the opposite vertices. Similarly, the Nagel point of  $ABC$  is the point of concurrence of line segments connecting the points where the excircles touch the sides of  $ABC$  with the opposite vertices (Figure 17, right)<sup>10</sup>.

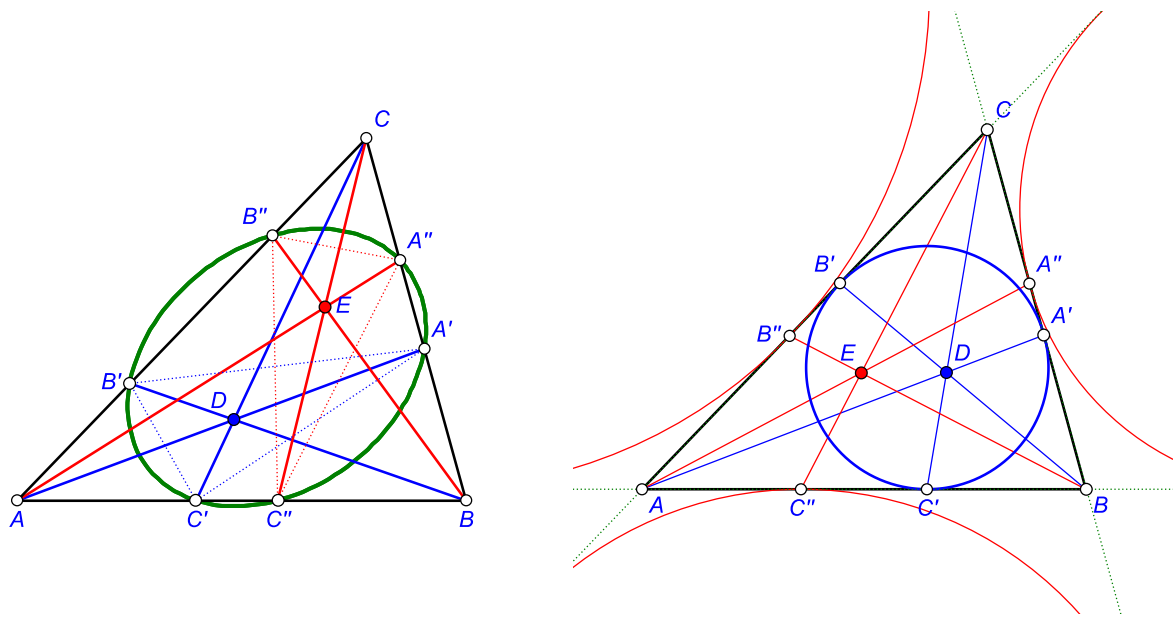


Figure 17

Here is a quick and efficient way of executing this construction<sup>11</sup>:

Steps	Commands	Comment
1. Draw a triangle ABC	<b>Point/Point</b> (for A,B,C) <b>Line/Polyline</b> (to connect A-B-C-A) <b>Special/Reference triangle</b>	Obtain a reference triangle.
2. Position the Gergonne point D and the Nagel point E of ABC	<b>Special/Triangle centres/Advanced/Gergonne point</b> <b>Special/Triangle centres/Advanced/Nagel point</b>	Keep the check-mark for using the reference triangle, since we construct the Gergonne and Nagel points of a previously

<sup>9</sup> OKExamples\OKG\_Plus\Triangle\_05.p

<sup>10</sup> OKExamples\OKG\_Plus\Triangle\_04.p

<sup>11</sup> OKExamples\OKG\_Plus\Triangle\_06.p

		defined reference triangle ABC.
3. Construct the Ceva-triangles of D and E	<b>Special/Triangle point derived objects/Triangle/Cevian</b> (choose then points D and E)	
4. Construct the ellipse and the characteristic points of the ellipse	<b>Circle/Conic 5 pts Point/Conic points</b>	Label the centre of the ellipse as S. (Figure 18, left.)
5. Inspect the centre S of the ellipse	<b>Special/Analyse object wrt. triangle</b> (pick S)	OK Geometry observes that S is the ETC centre X5452. (See Figure 18, right.)  A more detailed inspection with Triangle analysis suggest that, for example:  S = Isogonal conjugate of the Cross conjugate of X6 and D, or S = Complement of the Isogonal conjugate of X1486, or S = Complement of the Isotomic conjugate of (X3434)
6. Inspect the points on the constructed ellipse	<b>Special/Analyse object wrt. triangle</b> (pick the ellipse)	OK Geometry spots 3 centres along the ellipse (X3022, X3271, X4904). The extended search adds to this list 6 more transformed centres.

Here is an even more effective construction.

Steps	Commands	Comment
1a. Draw a triangle ABC	<b>Special/A, triangle</b>	Obtain a reference triangle ABC.
2a. Position the Gergonne point D and the Nagel point E of ABC	<b>Special/Objects by triangle centres</b> In the entry write 7,8	Since X7 is the Gergonne point, X8 is the Nagel point of the triangle.
3a. Construct the Ceva-triangles of D and E	<b>Special/Triangle point point derived objects/Conic/Bicevian conic</b> (choose then points 7 and 8)	

4a. Construct the characteristic points of the ellipse	<b>Point/Conic points</b>	Label the centre of the ellipse as S.
Proceed as in steps 5 and 6 of the previous construction.		

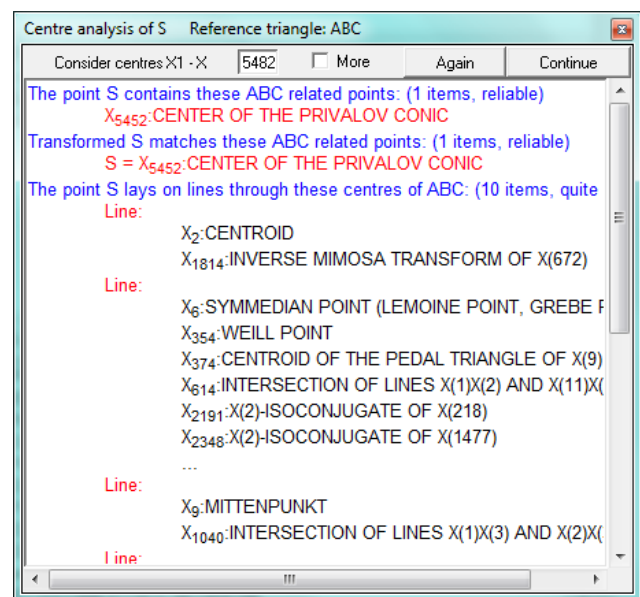
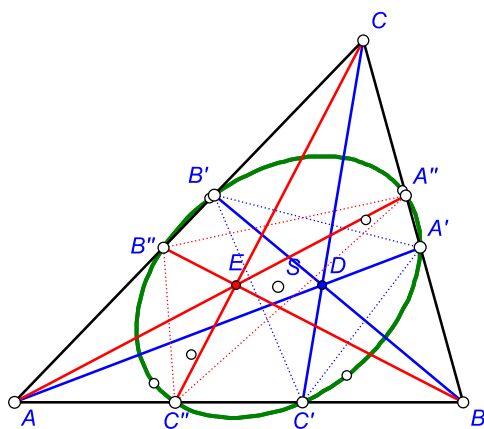


Figure 18

### Example 2 - Circum-side-circle

Given is a triangle ABC. A circle that has a side of ABC as its diameter is called a side-circle of ABC. So each triangle has three side-circles. We are looking for a way to construct the circle that touches internally all three side-circles of ABC, hereby called the circum-side-circle of ABC (Figure 19)<sup>12</sup>.

<sup>12</sup> OKExamples\OKG\_Plus\Triangle\_07.p

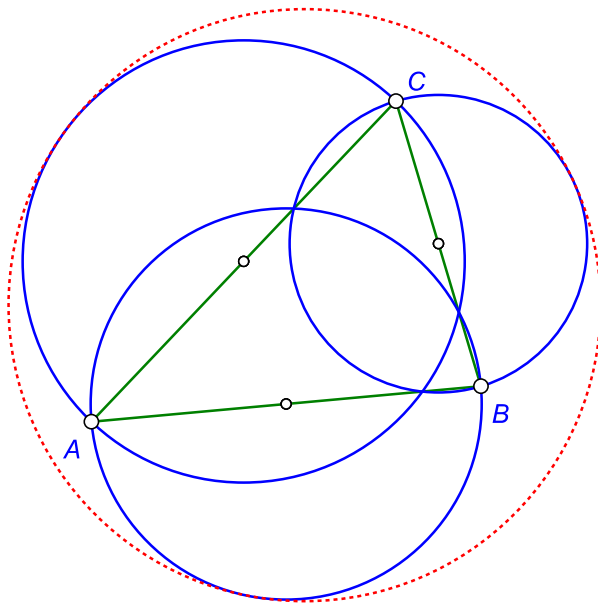


Figure 19

The strategy in the construction below is to find the three points of contact of the circum-side-circle with the three side-circles.

Steps	Commands	Comment
1. Draw a triangle ABC	<b>Point Point</b> (for A,B,C) <b>Line Polyline</b> (to connect A-B-C-A) <b>Special Reference triangle</b>	Try to make ABC visibly scalene.
2. Construct the three side-circles and the circum-side-circle	<b>Point Midpoint </b> (pick ABC) <b>On each segment</b> <b>Circle Circle centre+point</b> <b>Circle Circle 3 objects</b>	Use repeatedly the Alt button (in the editor's menu bar to obtain the desired circle that is tangent to three objects.
3. Construct the three points of tangency A', B', C' of the circum-side-circle	<b>Point Point</b>	The <b>Observe</b> command reveals that AA', BB', CC' concur in a point, say, P.
4. Construct the lines AA', BB', CC' and the concurrency point P.	<b>Line Line 2 pts</b> <b>Point Intersection</b>	

5. Inspect the properties of the point P in reference triangle ABC	Set ABC as the reference triangle. Enter the Triangle analysis module. Execute the Object analysis on the point P.	
6. Obtain information about the Paasche point	In the Triangle analysis module results place the cursor on X1123 and click the WhatIs button.	To obtain the basic information about the Paasche point place the cursor on this term and right- click the <b>WhatIs</b> command (Figure 20).

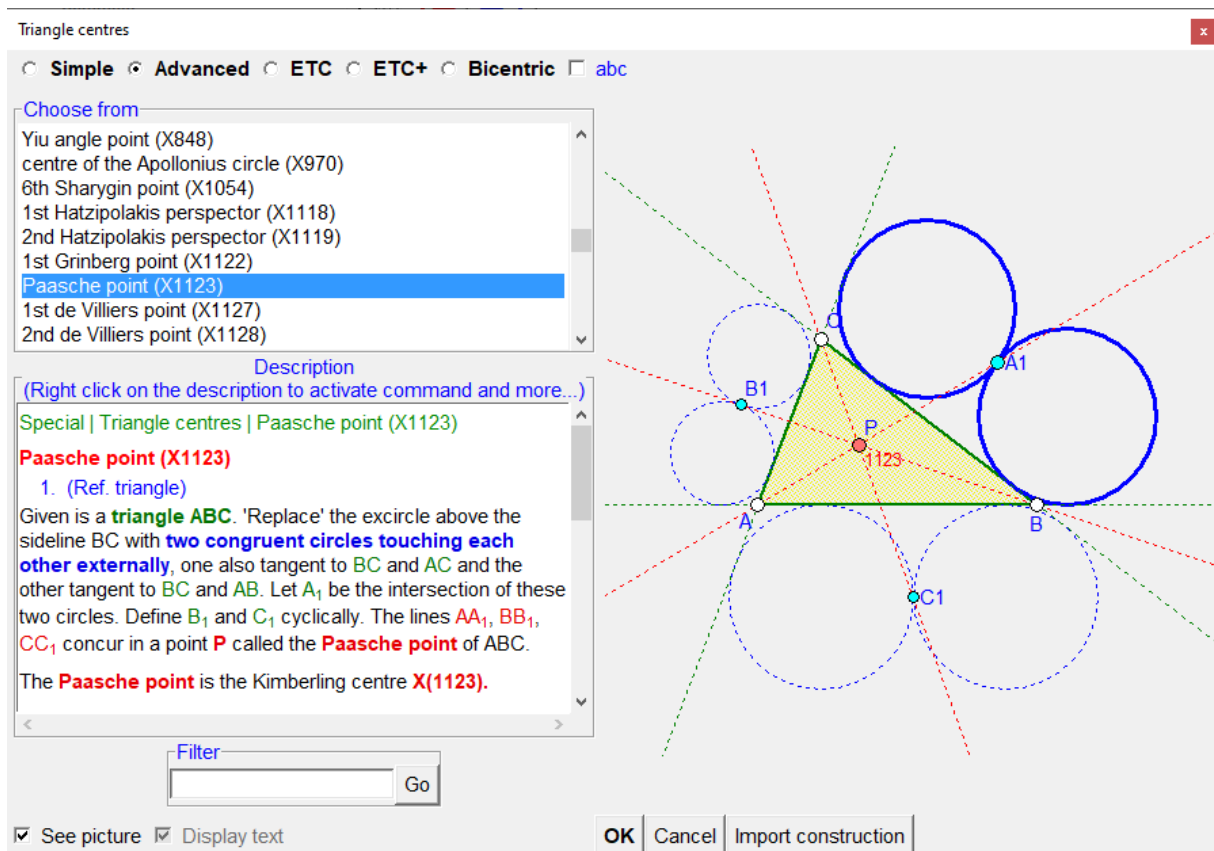


Figure 20

The Paasche point P of ABC can be constructed (using a homothety on a 'false' construction), and this gives way to the construction of the circum-side-circle. The construction steps are illustrated in Figure 21. A much more elegant way can be inferred with the Advanced query method described in Section 2.7.

The correctness of the construction via the Paasche point can be verified with algebraic means.

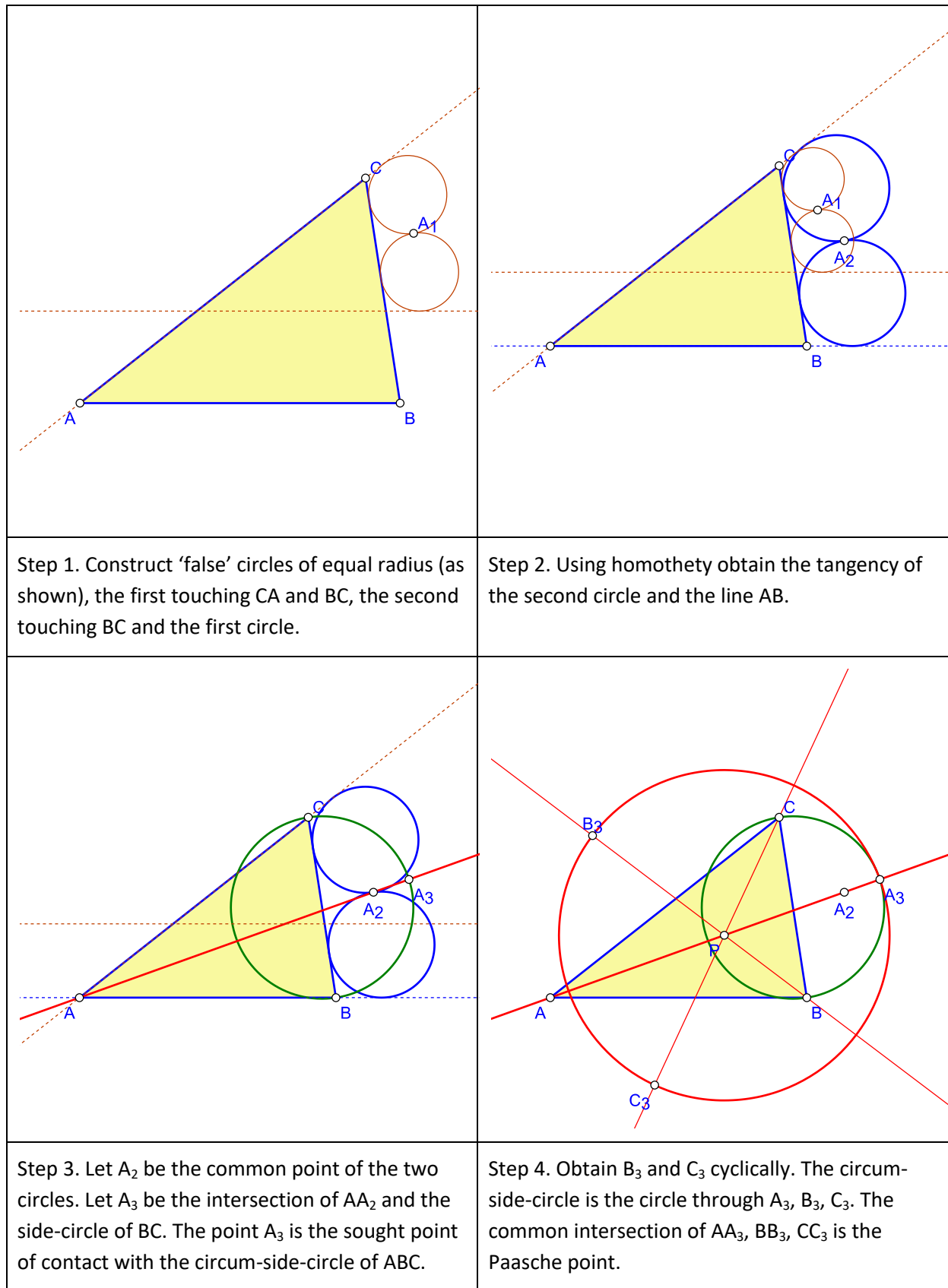


Figure 21

The circum-side-circle (Figure 19) is one of the 8 circles that are tangent to the three side-circles, see Figure 22<sup>13</sup>. Here are some of the observed properties:

One of the 8 circles contains all three side-circles, this is the circum-side-circle with centre  $P_0$  – the complement of the equal detour point  $X(176)$ .

Three of the 8 tangent circles are contained in exactly two of the side-circles, their centres are  $A_2, B_2, C_2$  (see Figure 22). The lines  $AA_2, BB_2, CC_2$  concur at point  $P_2$  – the Yiu-Pasche point  $X(1659)$ .

Three of the 8 circles are contained in exactly one of the side-circles, their centres are  $A_1, B_1, C_1$  (see Figure 22). The lines  $AA_1, BB_1, CC_1$  concur at point  $P_1$  – the isogonal conjugate of the 3rd Kenmotu homothetic centre  $X(5414)$ .

One of the 8 circles is contained all three side-circles. Its centre is  $P_3$  – the complement of the isoperimetric point  $X(175)$ .

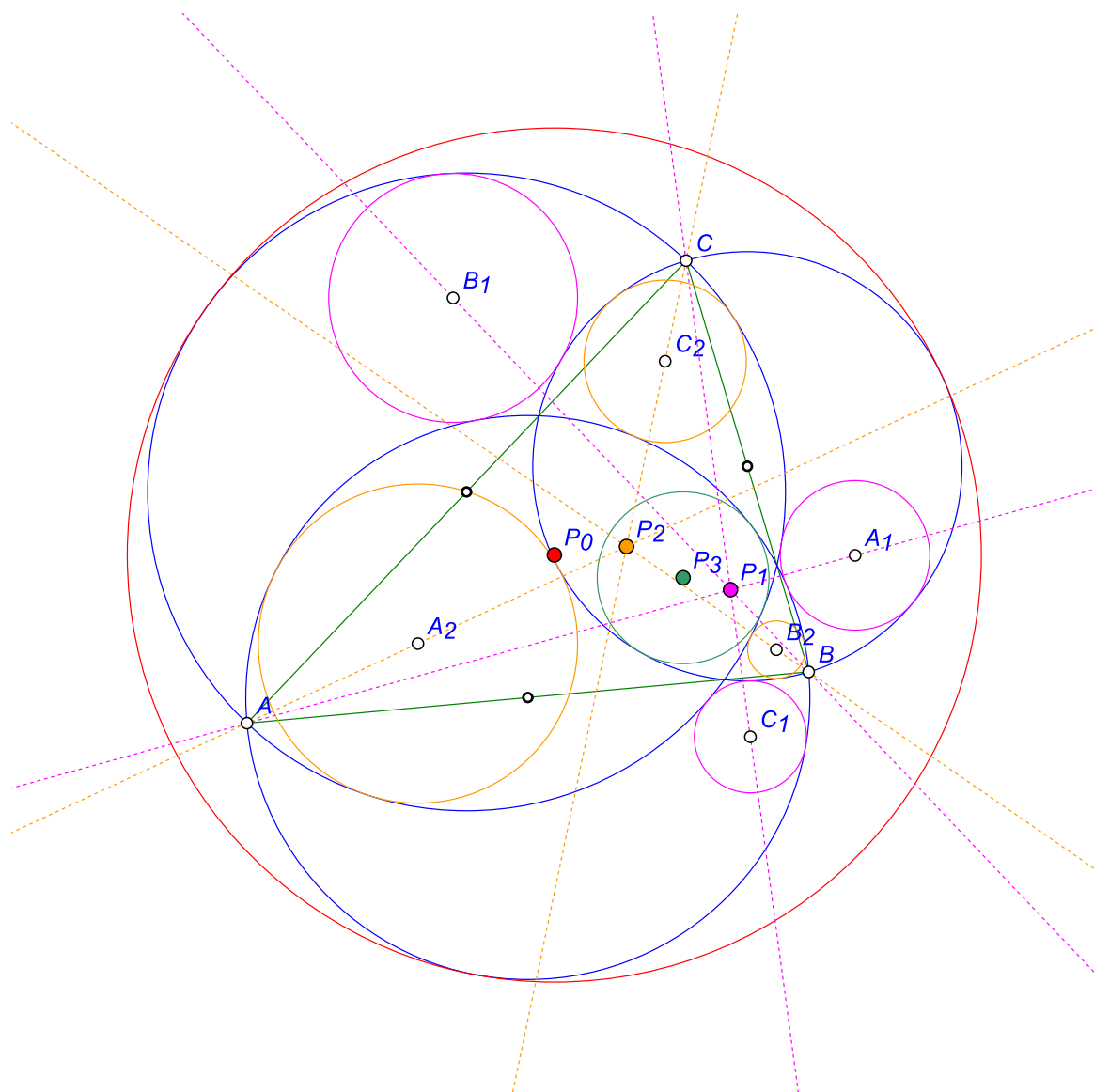


Figure 22

<sup>13</sup> OKExamples\OKG\_Plus\Triangle\_08.p

### Example 3 - Trisecting triangle perimeter by pedal triangle

Given is a triangle ABC and a point D. The pedal triangle of D in ABC has as its vertices the projections A', B', C' of the point D onto the sidelines of ABC. We are looking for a point D with the property that A', B', and C' cut the perimeter of ABC into three pieces of equal length (Figure 23), so that  $|B'A| + |AC'| = |C'B| + |BA'| = |A'C| + |CB'|$ .<sup>14</sup>

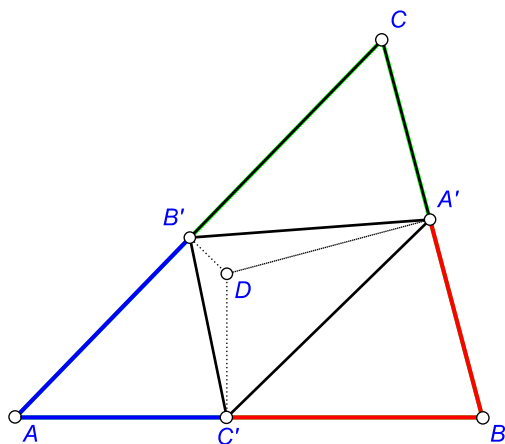


Figure 23

The following steps lead to a reliable hypothesis for the sought position of point D. The strategy is to (1) construct a general triangle, a point D and the corresponding pedal triangle, (2) redefine D with an implicit construction, (3) analyse the new position of the point D, (4) make an improved check for the proposed position.

Steps	Commands	Comment
1. Draw a triangle ABC and a point D	<b>Point/Point</b> (for A,B,C) <b>Line/Polyline</b> (to connect A-B-C-A) <b>Special/Reference triangle</b> <b>Point/Point</b> (Construct a free point D.)	Try to make ABC visibly scalene.
2. Construct the pedal triangle of D in ABC.	<b>Special   Triangle Point derived objects   Triangle   Pedal</b> (Pick D.) <b>Action   Labels   Label vertex</b> (Label the vertices A', B', C'.)	Note that ABC was previously set as a reference triangle.

<sup>14</sup> OKExamples\OKG\_Plus\Triangle\_09.p



3. Construct the obtained 'pieces of perimeter' as polylines and measure their length.	<b>Line / Polyline</b> (Pick B',A,C'; repeat the command for C',B,A' and A',C,B'.) <b>Number / Length perimeter</b> (Select the three constructed pieces.)	Name the lengths of the pieces as Len_A, Len_B, Len_C.
4. Check the sought condition	<b>Advanced / Check / Equivalence / SameValues3</b> (Pick the values Len_A, Len_B, Len_C.)	Name the check as SameValues3. Obviously, it turns out as False.
5. Implicit (re)construction of D.	<b>Advanced / Implicit construction</b> (Uncheck the restriction (OK); then pick the condition SameValues3 (OK) and the point D.)	The implicit construction should turn the condition to True.
6. Analyse the position of the point D	<b>Special / Analyse object wrt. triangle</b> (Pick D. ) OK observes that D is the centre X(165) – the centroid of the excentral triangle. To obtain a description of terms used in the result, position the cursor on it, right-click and use the command <b>WhatIs</b> .	Note that most items in the results are marked as 'unreliable', or even 'great caution'. This happens when the construction is based on optimisation or implicit construction.
7. Check the construction directly	<b>Action / Delete</b> (pick the point D) <b>Special / Triangle centres / ETC / X165</b> Repeat the steps 2-4.	Note that the check is now much more accurate since there are no implicit points in the construction.

## 2.6 Advanced triangle analysis

The **Triangle analysis** module is an advanced observation tool for triangle objects. The **Triangle analysis** module relates (by observation) examined objects to the characteristic objects of the reference triangle. In the observation analysis, the module considers the ETC centres of the reference triangle, various specific triangle transformations, characteristic lines, circles, and conics of the reference triangle. The module is an extension of the Simple observation of triangle objects (Section 2.2).

The module performs three, actually four types of analysis:

- **Object analysis** relates points, lines, circles, conics, cubics, and triangles to the characteristic points (centres), lines, circles, conics, cubics, and triangles of the reference triangle. Finds, for

example, that the isogonal conjugation of the examined point  $P$  lays on the Kiepert hyperbola of the reference triangle.

- **Two triangle centres** matches the centres of the examined triangle with the centres of the reference triangle. For example, finds that the centre  $X_{541}$  of the triangle  $ABC$  coincides with the centre  $X_{54}$  of the triangle  $PQR$ .
- **Rule-point analysis** matches points constructed from triangle centres (of some triangle) with the centres of the reference triangle. For example, for each ETC centre  $X_n$  it finds whether the midpoint of  $X_1$  and  $X_n$  is a known centre of the reference triangle.
- **2-rule-point analysis** matches points constructed from pairs of triangle centres (of some triangle) with the centres of the reference triangle. For example, for each pairs of ETC centres  $X_m, X_n$  it finds whether the midpoint of  $X_m$  and  $X_n$  is a known centre of the reference triangle.
- **Triangle perspectivity analysis** observes whether a given triangle is perspective to one of the characteristic triangles of the reference triangle. It considers various types of perspectivity and also analyses the centres of perspectivity.
- **Object analysis of generic constructions** performs an automated analysis on a family of dynamic constructions. We will explain this type of analysis in Section 5.5.

Of the four types, Object analysis is the most commonly used.

The best way to learn how to use the module is to use examples. Since all types of analysis have the same (or similar) parameters, modes of operation, and working procedures, we will first explain them. Then we will present some essential examples of the analysis.

### 2.6.1 Accessing the Triangle analysis module

To access the Triangle analysis module, click on the triangle ( $\Delta$ ) **symbol on the main menu bar**. Note that for a given construction the analysis can be performed only for objects that can be identified by labelled points on them. In particular, the vertices of the reference triangle should be labelled. Furthermore, the points, lines, circles, and conics to be analysed should contain (at least) 1,2,3,5 labelled points respectively.

### 2.6.2 Triangle analysis scheme

The best way to learn how to use triangle analysis is by using it. For this purpose some simple and advanced examples are provided (see Section 2.5).

The screenshot shows the 'Observe' tab in the OK Geometry Plus software. The interface is divided into several sections:

- Task Bar:** Contains buttons for 'Task', 'Sketch', 'Observe' (active), 'f()', a triangle icon, 'Project', and 'Report'.
- Type of analysis:**
  - Object analysis:** Selected with a radio button.
  - Two triangles centres:** Unselected.
  - Rule-points analysis:** Unselected.
  - 2-Rule-points analysis:** Unselected.
- Options:**
  - Transformed centres:** Unchecked checkbox.
  - Perspectivity centres:** Unchecked checkbox.
  - Short centre names:** Unchecked checkbox.
  - Searched properties:** Button labeled 'Auto'.
  - Report size:** Button labeled 'Short'.
- Data for analysis:**
  - Examined object(s):** Empty text field.
  - Reference triangle:** Text field containing 'ABC'.
  - Additional ref. points:** Empty text field.
  - Centres X1 - X:** Text field containing '16342+bic'.
- Observed results:** A large empty text area for displaying results.
- Bottom Toolbar:** Contains buttons for 'Exit', 'Study', 'Clear', 'AddToNotes', 'Show', 'WhatIs', and 'AddToComment'.

Figure 24

The following description refers to Figure 24, which shows the form that appears on the left-side pane during the Triangle analysis. Note that, depending on the construction, some of the fields may be missing or may be disabled. We provide here a detailed description of the form used in the module.

Entry	Description
<i>Object analysis</i>	Here you select the type of analysis you want to perform.
<i>Two triangles centres</i>	The default and most often used is the Object analysis.
<i>Rule-points analysis</i>	The analysis starts after you press the <b>Study</b> button.
<i>2-rule-points analysis</i>	

<p><i>Examined objects</i></p>	<p>In the entry, write the object(s) you want to examine in the analysis. Refer</p> <ul style="list-style-type: none"> <li>- to points as they are labelled, e.g. A' or B or C3,</li> <li>- to lines by specifying two points on them, e.g. A'B,</li> <li>- to circles by specifying three points on their circumference, e.g. A'BC3,</li> <li>- to conics by specifying 5 points on them, e.g. A'BC3DE.</li> </ul> <p>Multi-letter labels of vertices must be put in brackets, e.g. A(PQ3)B.</p> <p><b>Note.</b> You can also specify two or more objects to be examined. Separate them with space or comma, e.g., A E'. In this case, a sequence of analyses is performed, one for each Examined object.</p>
<p><i>Reference triangle</i></p>	<p>Specify the reference triangle by naming its vertices, e.g. ABC. Analysis will use the centres and characteristic objects of the reference triangle when studying the examined object(s).</p> <p><b>Note. Avoid reference triangles that are isosceles or almost isosceles.</b> In such triangles, many centres are unacceptably close together, so that OK Geometry refuses to perform the analysis.</p> <p><b>Note.</b> It is possible to specify two or more reference triangles (separated by spaces or commas, e.g. ABC D1E'F). In this case , a sequence of analyses is performed, one for each specified reference triangle.</p>
<p><i>Additional reference points</i></p>	<p>Here you can specify (optional) points to be considered in the analysis, in addition to the ETC centres of the reference triangle.</p>
<p><i>Condition parameter</i></p>	<p>The entry appears only if the considered construction contains one or more condition parameter.</p> <p>If a condition is selected in this entry, the analysis will be executed only if (or when) that condition is met, i.e. is true.</p> <p>Selecting a condition can be important when the studied construction</p> <ul style="list-style-type: none"> <li>- is based on an implicit construction, or</li> <li>- is based on the optimisation of a parameter, or</li> <li>- is a generic construction.</li> </ul> <p>In such situations, it may occur that the construction fails to succeed, so the analysis cannot or should not be performed. In the Sketch Editor you can define a condition (check) that is true if the analysis can be performed. Then use the condition as a condition parameter for the triangle analysis.</p>
<p><i>Centres X1-X</i></p>	<p>Here you specify the highest index of ETC triangle centres to be considered in the analysis. The more centres considered, the longer the computation will take and the more extensive list of results will be. For the most exhaustive analysis choose 16342+bic. Note that the when the index greater than 16342 is chosen, the analysis is less reliable and not all tests will be performed.</p>

<i>Transformed centres</i>	<p>Select this option if you want in the analysis to consider different triangle transformations of the ETC centres of the reference triangle (or the examined points)..</p> <p>When this option is selected, the analysis finds facts like ‘an examined point is the cyclocevian of X68’, or ‘the complement of the examined point is the isotomic conjugate of X25’.</p>
<i>Perspectivity centres</i>	<p>This option is valid only when the examined object is a triangle.</p> <p>By default, the object analysis of a triangle checks whether the examined triangle is in any way perspective (perspective, orthologic, parallelogic, cyclogic) to characteristic triangles of the reference triangle. If this option is ON, all the found perspectivity centres are calculated and stored, so that they can be retrieved and examined.</p> <p>For example, let us examine (with the option Perspectivity centres ON) the extouch triangle A'B'C' of the reference triangle ABC. After the Study button is pressed, OK Geometry reports the observation of 39+92 cases of perspectivity of the triangle A'B'C' with various triangles related to the reference triangle ABC. These cases give rise to 198 centres of perspectivity (not all of which differ from each other) denoted by %1 to %198. The report says, for example, that the triangle A'B'C' is related to the Garcia inner triangle in the following way:</p> <p style="text-align: center;">perspective (%59), orthologic (%60, %61).</p> <p>Thus %59 is the perspectivity centre of A'B'C' and the Garcia inner triangle of ABC, while the point %60 is the orthological centre of A'B'C' and the Garcia inner triangle of ABC, and the point %61 is the orthological centre of the Garcia inner triangle of ABC and A'B'C'.</p> <p>To learn more about the point %60, simply write %60 as the examined object and study it. The notation %% denotes all points %1-198 without repetition.</p>
<i>Short centre names</i>	<p>When this option is selected, the descriptive part of the triangle centre names are omitted from the report.</p> <p>For example, if this option is checked, in the list of results the Nagel point will be referred to as X8 instead of X8: NAGEL POINT</p>
<i>Searched properties</i>	<p>The default option is <b>Auto</b> (i.e., all properties are included). If you click the Auto button, you can specify a <b>selection of</b> properties to be included in the triangle analysis.</p> <p>Limiting the triangle analysis to one or a few properties is particularly useful when performing the triangle analysis for a generic construction.</p>

<p><i>Report size</i></p>	<p>For the Object analysis the report size which can be</p> <ul style="list-style-type: none"> <li>- Minimal (only a few facts of the very basic properties are listed),</li> <li>- Short (the default option, a limited number of basic properties are listed, only a limited number of triangle triangles and triangle circles are considered),</li> <li>- Extended (only the first 100 facts of all considered properties concerning all triangle objects are listed)</li> <li>- All (all facts of all properties are listed; may result in a very long list of facts).</li> </ul> <p>For the Two triangle centres analysis and for the Rule-points analysis the report can be</p> <ul style="list-style-type: none"> <li>- Minimal (only the first few results are listed),</li> <li>- Short (only the first few results are listed),</li> <li>- Extended (the first 1000 results are listed),</li> <li>- All (all found facts are listed).</li> </ul> <p>For the Rule-points analysis the report can be</p> <ul style="list-style-type: none"> <li>- Minimal (refers to the first 20 examples of the ETC1 rule),</li> <li>- Short (refers to the first 100 examples of the ETC1 rule),</li> <li>- Extended (refers to the first 1000 examples of the ETC1 rule),</li> <li>- All (refers to all examples of the ETC1 rule).</li> </ul>
---------------------------	--

### 2.6.3 Observed results

1. Once you completed the schema in Figure 24, you can start the analysis by pressing the **Study** button. The reported results of the analysis are displayed in the **Observed results** section. Figure 25 shows an illustrative example of part of the displayed results (the analysed object is a point D).
2. The analysis can be performed several times in sequence. Simply enter the data, set the options, and press the **Study button**.
3. The reported results are always appended to the previous content of the section Observed results. At any time, e.g. before a new analysis is run, you can clear the content of the section Observed results with the **Clear** button.
4. The reports can be freely edited.

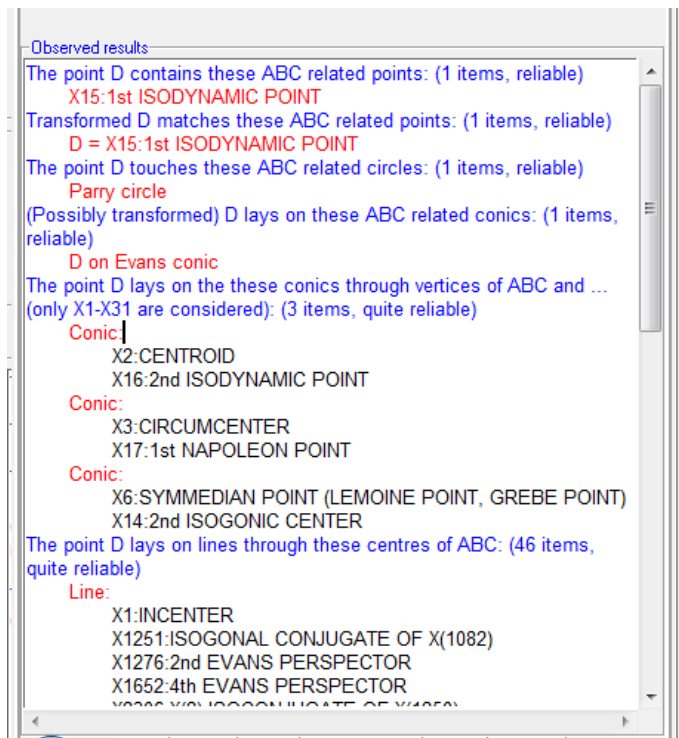


Figure 25

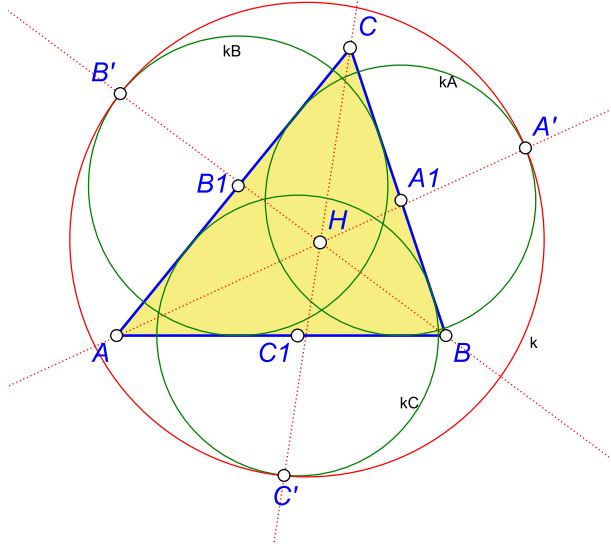
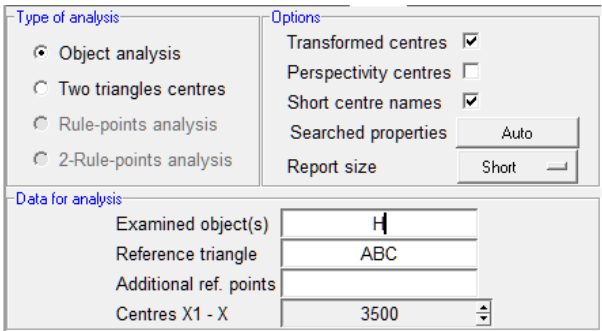
5. If you want to include a part of the Observed results in the OK Geometry report (i.e. the printable report of the project) for the studied construction, put this part of the results in a block and press the **AddToReport** button.
6. You can append a part of the Observed results to the comment of the current construction. To do this, put this part of the results in a block and press the **AddToComment** button.
7. To get an explanation of objects mentioned in the Observed results section, click on the **WhatIs** button. The cursor then takes the form of a question mark with an arrow. If you click with the arrow on a term in the Observed result section, you will get an explanation of that term (sometimes the neighbouring words are also considered). For example, to find out the definition of the Parry circle or of  $X_{17}$ , press the **WhatIs** button and click with the arrow of the question-cursor on 'Parry' or on ' $X_{17}$ '.
8. To visualise on the construction one or more objects that are mentioned in the Observed results section, position the cursor on the name of the object to be shown or put in a block the objects to be shown. Then press the **Show** button. For example, to visualise  $X_{16}$ , position the cursor on  $X_{16}$  and press the **Show** button. To visualise the conic through  $X_2$ ,  $X_{16}$ , and the vertices of the reference triangle, put in block the three lines from Conic to  $X_{16}$ , and press the **Show** button.
9. The **Glossary help** is accessible from the context menu when you **right-click on the Observed results** section. The command turns the cursor into a spider. A click of the spider on a term in the Observed results section activates the glossary for that term.
10. To exit from the Analysis module, press the **Exit** button or press the ( $\Delta$ ) button in the main menu bar.

### 2.6.4 Object analysis

The Object analysis of the Triangle analysis module observes the relationship between the examined objects and the characteristic objects of the reference triangle (ETC centres, characteristic lines, circles, conics, cubics). Essential data for the Object analysis is the reference triangle, named after its vertices, e.g. ABC. Essential is also one or more objects to be examined. They are identified by labelled points on them: one label refers to points (e.g. P), two labels to lines (e.g. PQ), three labels to circles and triangles (e.g. PQR), five labels to conics (e.g. PQRST), six labels to cubics through the vertices of the reference triangle (e.g., PQRSTU), nine vertices to cubics through nine points (e.g., PQRSTUVZW). By filling in the **Examined objects** entry with 'P Q PQR' we run the analysis on the points P and Q, the circle through P,Q,R and the triangle PQR.

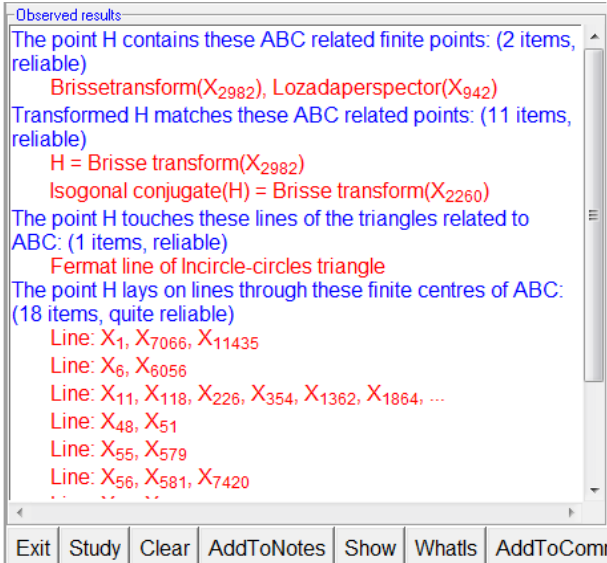
#### Example

Here is an illustrative example of triangle analysis<sup>15</sup>.

	<p>Given is a triangle ABC. Let <math>k_A</math> be the circle touching the baselines AB and AC and having centre <math>A_1</math> on side BC. Define <math>B_1</math>, <math>C_1</math> and <math>k_B</math>, <math>k_C</math> cyclically.</p> <p>Let the circle <math>k</math> touch internally the circles <math>k_A</math>, <math>k_B</math>, <math>k_C</math> in points <math>A'</math>, <math>B'</math>, <math>C'</math>. OK Geometry observes that the lines <math>AA'</math>, <math>BB'</math>, <math>CC'</math> meet at a common point H.</p> <p>We analyse the point H in relation to the reference triangle ABC.</p>
	<p>Make sure that the <b>Type of analysis</b> is set to <b>Object analysis</b>.</p> <p>Select <b>Short centre names</b> if you prefer a compact report without explanatory names of triangle centres.</p> <p>Select <b>Transformed centres</b> if you want the transformations of H to be included in the analysis.</p> <p>It is advisable to initially select a <b>Short Report size</b> with <b>Transformed centres</b> OFF. Then, if</p>

<sup>15</sup> OKExamples\OKG\_Plus\Analysis\_01.p



	<p>needed, increase Report size (more instances will be displayed) and/or select Transformed centres (additional properties will be considered).</p> <p>Press <b>Study</b> to start the analysis.</p>
	<p>Here is an excerpt of the results(some irrelevant lines have been deleted).</p> <p>The observation shows that H is, for example, the Lozada perspector of X(942), that the isogonal conjugate of H appears to be X(2260), that H is the intersection of the lines X(1)X(7066) and X(6)X(6056), etc.</p> <p>To get an explanation of the Lozada perspector, first click on <b>WhatIs</b> button first and then click on the term Lozada perspector.</p> <p>To visualise objects from the report, put them in a block and click the <b>Show</b> command.</p> <p>Other available functions are described in section 0.</p>

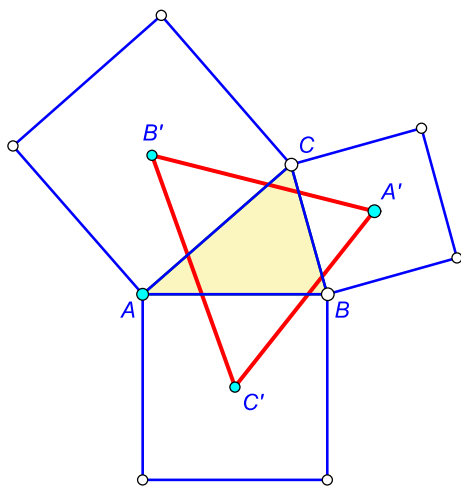
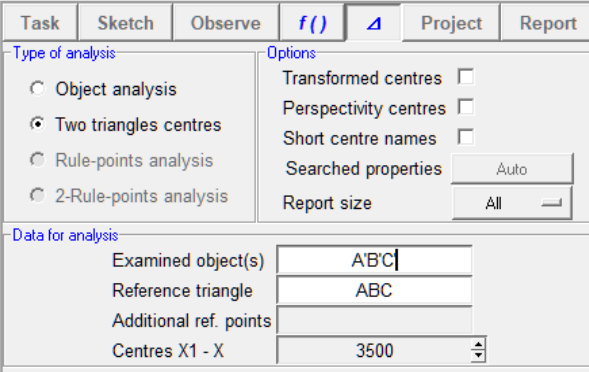
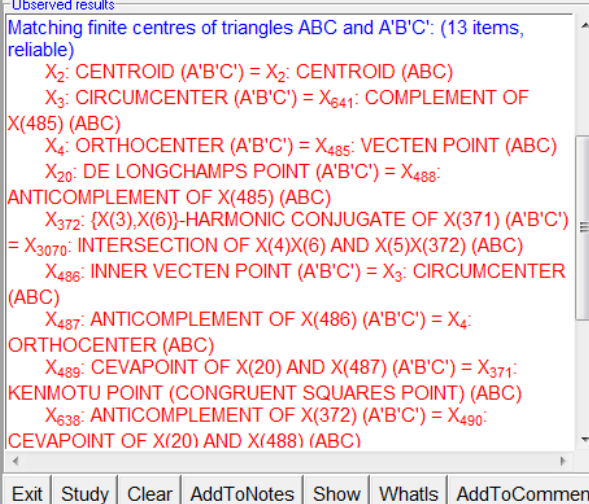
### 2.6.5 Two triangle centres

Two triangles centres analysis matches the centres of a triangle to the centres of the reference triangle.

#### Example

Here is an illustrative example how to perform such analysis.<sup>16</sup>

<sup>16</sup> OKExamples\OKG\_Plus\Analysis\_02.p

	<p>Given is a triangle ABC. Consider the outward squares on the sides BC, CA, AB of the triangle ABC. The centres <math>A'</math>, <math>B'</math>, <math>C'</math> of these squares are the vertices of the outer Vecten triangle of ABC.</p> <p>With <b>Two triangles centres</b> analysis we examine whether any centre of <math>A'B'C'</math> coincides with any centre of ABC.</p>
	<p>Select <b>Two triangles centres</b> as Type of analysis.</p> <p>We choose All for <b>Report size</b>, otherwise only the first few results will be displayed.</p> <p>In our analysis only the first 3500 centres are considered. If you want to consider also the transformations of the centres (e.g. isogonal conjugation, isotomic conjugation), checkmark the <b>Transformed centres</b> option.</p>
	<p>Here is the list of observed matches. If we select the <b>Transformed centres</b> option and/or increase the number of considered centres, a much longer list will appear.</p>

## 2.6.6 Rule-points and 2-rule-points analysis

Rule-points and 2-rule-points analysis are quick methods of checking whether points constructed from the ETC centres of a triangle coincide with any ETC centre of the reference triangle. **Rule-point**

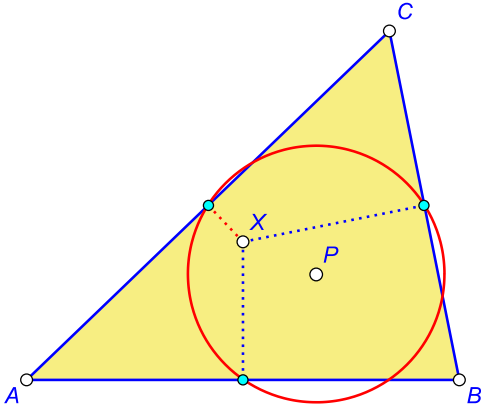
**analysis** is performed on a generic construction, where a point, say  $X$ , is created with the **ETC1 rule** (for some triangle), and there is also a point, say  $P$ , which constructionally depends on the point  $X$ . The point  $X$  can occupy the position of various ETC centres of this triangle. The rule-centres analysis of the point  $P$  finds out whether the point  $P$  coincides with a centre of the reference triangle as the point  $X$  takes the place of the ETC centres of a triangle. It is a rather simple and fast operation.

**2-rule-point analysis** is likewise performed on a generic construction, where a point, say  $X$ , is created with the **ETC1 rule** (for some triangle), a point, say  $Y$ , is created with the **ETC2 rule** and there is also a point, say  $P$ , which constructionally depends on the points  $X$  and  $Y$ . The points  $X$  and  $Y$  can independently occupy the position of various ETC centres of this triangle. The 2-rule-centres analysis of the point  $P$  finds out whether the point  $P$  coincides with a centre of the reference triangle as the points  $X$  and  $Y$  take the place of the ETC centres of a triangle.

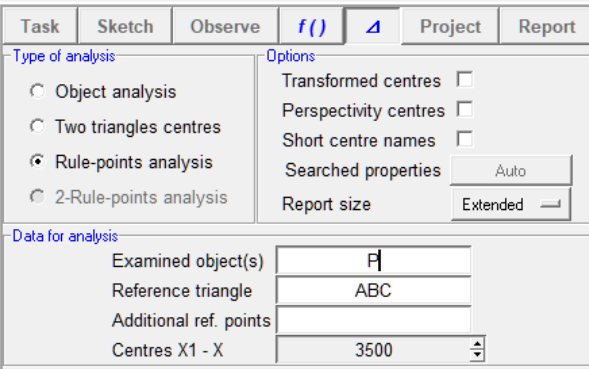
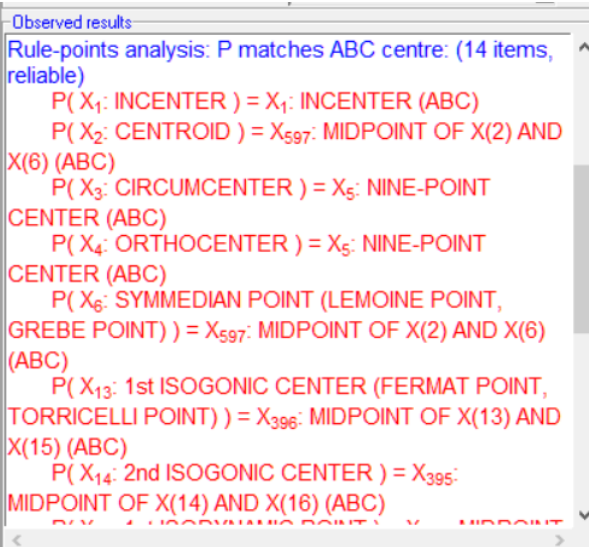
Please, refer to Section 5.3.5 for details on ETC1 rule and the generic constructions. Also note that the triangle analysis on generic constructions (Section 5.5) allows more complex (but not so quick) observations on generic triangles.

### Example

Here is an illustrative example how to perform a rule-points analysis<sup>17</sup>.

	<p>Given is a triangle <math>ABC</math>. For a point <math>X</math>, let <math>P</math> be the circumcentre of the pedal triangle of <math>X</math> in <math>ABC</math>. We investigate whether <math>P</math> is a known centre of <math>ABC</math> if <math>X</math> takes the position of the first 1000 triangle centres of <math>ABC</math>.</p> <p>In the generic construction we define the point <math>X</math> by the ETC1 rule on the triangle <math>ABC</math> (i.e., <math>X</math> is a 'variable triangle centre' of <math>ABC</math>).</p> <p>We construct the circle by projecting <math>X</math> onto the sidelines of <math>ABC</math>. (Note that <math>A'B'C'</math> it is also available as the pedal circle of a point in a triangle).</p> <p>Now switch to the Triangle analysis module.</p>
---	---

<sup>17</sup> OKExamples\OKG\_Plus\Analysis\_03.p

	<p>Select <b>Rule-points analysis</b> as a Type of analysis.</p> <p>We choose Extended for <b>Report size</b>, so that X takes the position of the first 1000 ETC centres of ABC.</p> <p>The positions of the point P are matched with the first 3500 centres of ABC. If also the transformations of these centres (e.g., isogonal conjugation, isotomic conjugation) should also be considered, select the <b>Transformed centres</b> option.</p>
	<p>Here is the list of matches. For example, we find that if X is the symmedian point of ABC then the centre P of the circle is the midpoint the centroid and the symmedian point of ABC.</p>

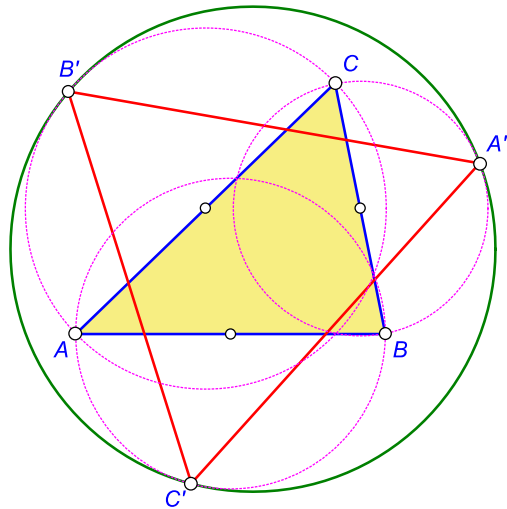
### 2.6.7 Triangle perspectivity analysis

With this analysis we investigate whether a triangle is perspective to different characteristic triangles of the reference triangle (e.g., the orthic triangle). The analysis considers up to 230 characteristic triangles of the reference triangle. For each triangle it is checked, whether it is perspective, parallelologic, orthologic or cyclologic to the examined triangle. In case of a positive result, the perspectivity centres are found – they are denoted as %1, %2, .... The set of all found perspectivity centres is denoted y %.

#### Example

Here is an illustrative example how to perform a triangle-perspectivity analysis<sup>18</sup>.

<sup>18</sup> OKExamples\OKG\_Plus\Analysis\_04.p



Given is a triangle  $ABC$ . Consider the three circles with sides  $BC$ ,  $CA$ , and  $AB$  as their respective diameters. Let  $A'$ ,  $B'$ ,  $C'$  be the points of contact of these circles with the circle that touches them all internally.

We examine if the triangle  $A'B'C'$  is perspective to any of the characteristic triangles of  $ABC$ .

Type of analysis		Options	
<input checked="" type="radio"/> Object analysis		Transformed centres	<input type="checkbox"/>
<input type="radio"/> Two triangles centres		Perspectivity centres	<input checked="" type="checkbox"/>
<input type="radio"/> Rule-points analysis		Short centre names	<input type="checkbox"/>
<input type="radio"/> 2-Rule-points analysis		Searched properties	Auto
		Report size	All

Data for analysis	
Examined object(s)	A'B'C'
Reference triangle	ABC
Additional ref. points	
Centres X1 - X	3500

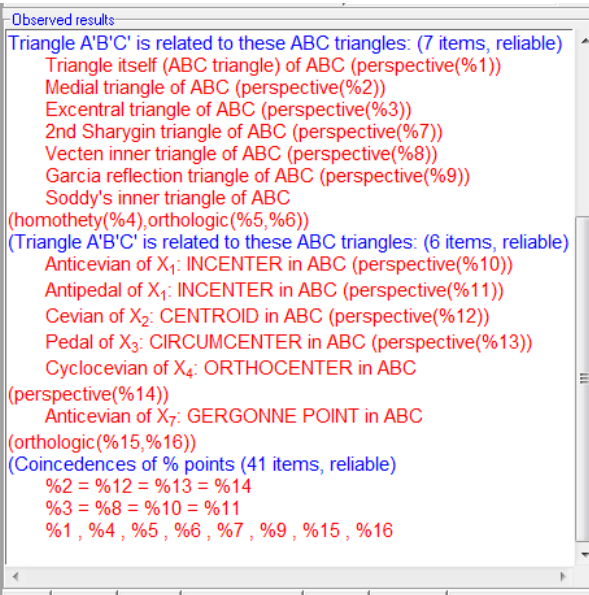
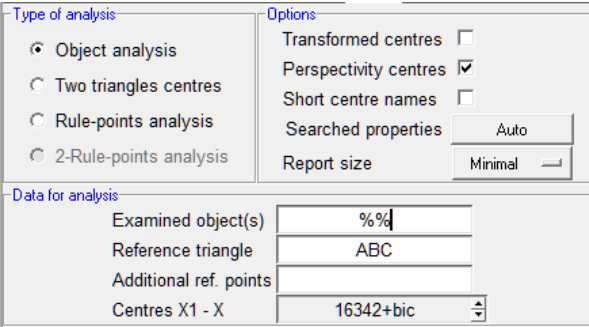
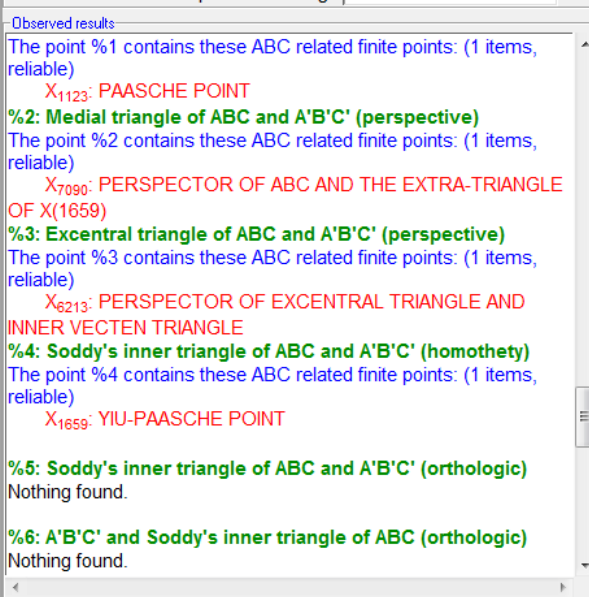
Select **Object analysis** as Type of analysis.

We select **All** for Report size, so that all related triangles will be considered and all the results will be displayed.

Select **Perspectivity centres**, so that the found perspectivity centres are stored as points %1, %2, ...

Our reference triangle is  $ABC$  and the examined triangle is  $A'B'C'$ . (Note that its circumcircle of  $A'B'C'$  will be examined as well).

The options **Transformed centres** and **Considered centres** options will be taken into account later when analysing the %points.

 <p>Observed results</p> <p>Triangle A'B'C' is related to these ABC triangles: (7 items, reliable)</p> <ul style="list-style-type: none"> <li>Triangle itself (ABC triangle) of ABC (perspective(%1))</li> <li>Medial triangle of ABC (perspective(%2))</li> <li>Excentral triangle of ABC (perspective(%3))</li> <li>2nd Sharygin triangle of ABC (perspective(%7))</li> <li>Vecten inner triangle of ABC (perspective(%8))</li> <li>Garcia reflection triangle of ABC (perspective(%9))</li> <li>Soddy's inner triangle of ABC (homothety(%4),orthologic(%5,%6))</li> </ul> <p>(Triangle A'B'C' is related to these ABC triangles: (6 items, reliable)</p> <ul style="list-style-type: none"> <li>Anticevian of X<sub>1</sub>: INCENTER in ABC (perspective(%10))</li> <li>Antipedal of X<sub>1</sub>: INCENTER in ABC (perspective(%11))</li> <li>Cevian of X<sub>2</sub>: CENTROID in ABC (perspective(%12))</li> <li>Pedal of X<sub>3</sub>: CIRCUMCENTER in ABC (perspective(%13))</li> <li>Cyclocevian of X<sub>4</sub>: ORTHOCENTER in ABC (perspective(%14))</li> <li>Anticevian of X<sub>7</sub>: GERGONNE POINT in ABC (orthologic(%15,%16))</li> </ul> <p>(Coincences of % points (41 items, reliable)</p> <ul style="list-style-type: none"> <li>%2 = %12 = %13 = %14</li> <li>%3 = %8 = %10 = %11</li> <li>%1, %4, %5, %6, %7, %9, %15, %16</li> </ul> <p>Exit Study Clear AddToNotes Show WhatIs AddToComment</p>	<p>Here is the list of observed characteristic triangles of ABC, which are homothetic, perspective, parallelologic, orthologic or cyclogic to A'B'C'. At the same time, all the corresponding perspectivity centres are computed.</p> <p>The calculated centres are listed at the end of the report. Some of the centres coincide.</p>								
 <p>Type of analysis</p> <ul style="list-style-type: none"> <li><input checked="" type="radio"/> Object analysis</li> <li><input type="radio"/> Two triangles centres</li> <li><input type="radio"/> Rule-points analysis</li> <li><input type="radio"/> 2-Rule-points analysis</li> </ul> <p>Options</p> <ul style="list-style-type: none"> <li>Transformed centres <input type="checkbox"/></li> <li>Perspectivity centres <input checked="" type="checkbox"/></li> <li>Short centre names <input type="checkbox"/></li> <li>Searched properties <input type="button" value="Auto"/></li> <li>Report size <input type="button" value="Minimal"/></li> </ul> <p>Data for analysis</p> <table border="1"> <tr> <td>Examined object(s)</td> <td>%%</td> </tr> <tr> <td>Reference triangle</td> <td>ABC</td> </tr> <tr> <td>Additional ref. points</td> <td></td> </tr> <tr> <td>Centres X1 - X</td> <td>16342+bic</td> </tr> </table>	Examined object(s)	%%	Reference triangle	ABC	Additional ref. points		Centres X1 - X	16342+bic	<p>The following operation must be performed <b>before we delete the results with the Clear button</b>.</p> <p>We can examine the centres %1, %2, ... as ordinary points.</p> <p>On the left we show how to examine all of the perspectivity centres at once (%% means the set of all found centres). Note that Report size was set to <b>Minimal</b>, otherwise the computation of detailed results will take time.</p>
Examined object(s)	%%								
Reference triangle	ABC								
Additional ref. points									
Centres X1 - X	16342+bic								
 <p>Observed results</p> <p>The point %1 contains these ABC related finite points: (1 items, reliable)</p> <ul style="list-style-type: none"> <li>X<sub>1123</sub>: PAASCHE POINT</li> </ul> <p>%2: Medial triangle of ABC and A'B'C' (perspective)</p> <p>The point %2 contains these ABC related finite points: (1 items, reliable)</p> <ul style="list-style-type: none"> <li>X<sub>7090</sub>: PERSPECTOR OF ABC AND THE EXTRA-TRIANGLE OF X(1659)</li> </ul> <p>%3: Excentral triangle of ABC and A'B'C' (perspective)</p> <p>The point %3 contains these ABC related finite points: (1 items, reliable)</p> <ul style="list-style-type: none"> <li>X<sub>6213</sub>: PERSPECTOR OF EXCENTRAL TRIANGLE AND INNER VECTEN TRIANGLE</li> </ul> <p>%4: Soddy's inner triangle of ABC and A'B'C' (homothety)</p> <p>The point %4 contains these ABC related finite points: (1 items, reliable)</p> <ul style="list-style-type: none"> <li>X<sub>1659</sub>: YIU-PAASCHE POINT</li> </ul> <p>%5: Soddy's inner triangle of ABC and A'B'C' (orthologic)</p> <p>Nothing found.</p> <p>%6: A'B'C' and Soddy's inner triangle of ABC (orthologic)</p> <p>Nothing found.</p> <p>Exit Study Clear AddToNotes Show WhatIs AddToComment</p>	<p>Here is a snippet of the results. We see that X(1123) is the centre of perspectivity of ABC and A'B'C'. Use <b>What is</b> to learn about the Paasche point and <b>Show</b> to display it.</p> <p>Note that nothing about %5 is found. To get more observational data, enter %5 as Examined object, Increase <b>Report size</b> and perhaps checkmark <b>Transformed centres</b>.</p>								

## 2.7 Quadrilateral objects.

### 2.7.1 Basic concepts

The theory and classification of quadrilateral objects used in OK Geometry is based on vanTienhoven's text *Encyclopedia of Quadri-Figures*<sup>19</sup>. We use slightly different terminology, but the quadrilateral objects and their classification are the same as in his text.

We define a **quadrilateral** as a cyclic polyline with 4 vertices in general position. A quadrilateral therefore consists of 4 vertices, of which no three of them are collinear, and 4 sides. Each vertex is the end point of exactly 2 sides. There are exactly 4 pairs of contingent sides and exactly 2 pairs of non-contingent (opposite) sides. There are also exactly 4 pairs of vertices that share a common side and 2 pairs of opposite vertices (which have no common side). The lines through opposite vertices are called the diagonal lines. A quadrilateral can be self-intersecting (i.e. opposite sides can intersect) or not. In the later case, the region bounded by the sides may or may not be convex (Figure 26).

We use the terms quadrigon, quadrangle, quadrilateral as synonyms.

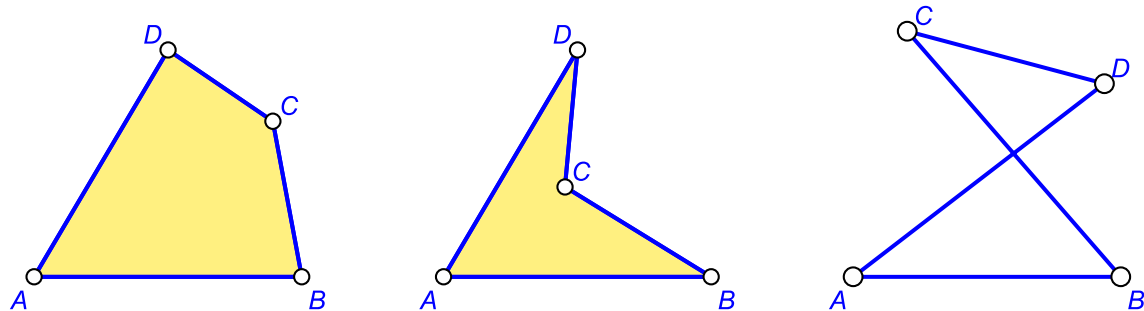


Figure 26

Quadrilaterals that have the same vertices but different sidelines are called **QA-siblings**. A quadrilateral ABCD has two QA-siblings: ABDC and ADBC (Figure 27). If the construction of an object from the vertices of a quadrilateral results in the same object for all QA-siblings, then this object is called a **QA-object**. For example, the centroid of the vertices of a quadrilateral is such an object. We refer to it as the QA-centroid (QA-P1). To avoid misunderstandings we always add to names the vanTienhoven's classification of the object, in our case QA-P1.

On the other hand, the intersection of diagonals of quadrilaterals is clearly not a QA-object.

<sup>19</sup> <https://chrisvantienhoven.nl/mathematics/encyclopedia>

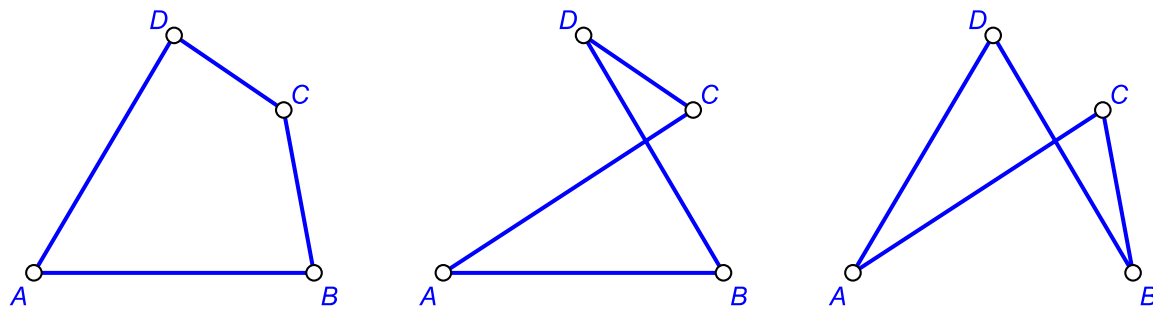


Figure 27

Quadrilateral that have the same side-lines but different vertices are called **QL-siblings**. A quadrilateral ABCD has 2 QL-siblings (Figure 28): AECF and BEDF, where  $E = AB \cap CD$  and  $F = BC \cap DA$ .

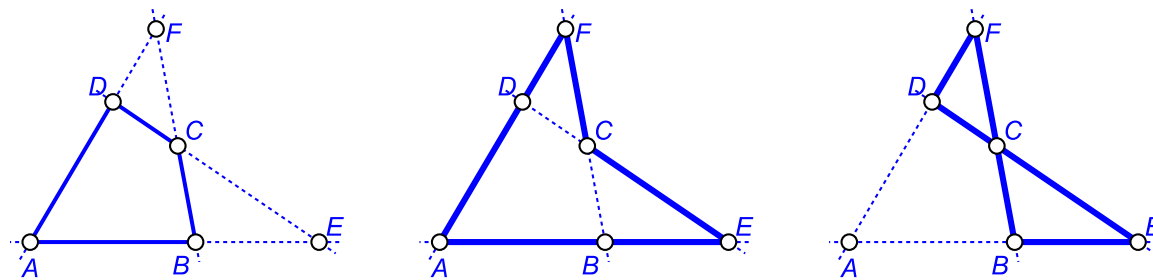


Figure 28

If the construction of an object from the side-lines of the quadrilateral results in the same object for all its QL-siblings, then this object is called a **QL-object**. For example, the Newton line of a quadrilateral (i.e. the line through the midpoints of the diagonals, Figure 29) is a QL-object. We refer to it as the QL-Newton line (QL-L1). To avoid misunderstandings we always add to names the vanTienhoven's classification, in our case QL-L1.

On the other hand, the intersection of diagonals is clearly not a QL-object.



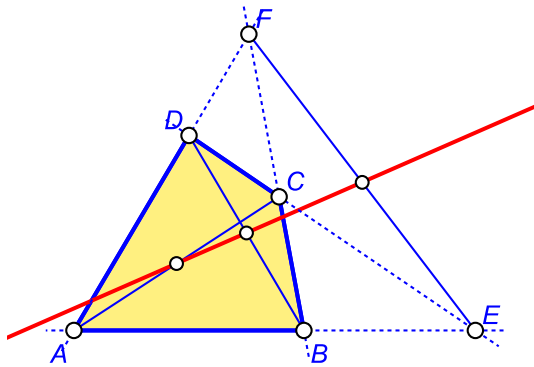


Figure 29

Finally, objects resulting from constructions that are invariant under cyclic permutation of the vertices (or sidelines) of a quadrilateral are called **QG-objects**. QG-objects are clearly also QL- and QA-objects. For example, the third diagonal of ABCD (line EF on Figure 29) is an example of a QG-object. We refer to it as QG-3rd diagonal (QG-L1). To avoid misunderstandings we always add to names the vanTienhoven's classification, in our case QG-L1.

### 2.7.2 Special commands for quadrilateral objects

The Sketch Editor in Plus mode has a subgroup of **Special** commands dedicated to quadrilateral objects. The commands in this group are similar to those for the triangle object and have a similar syntax.

You can use the commands in this subgroup to construct various objects that are related to a quadrilateral. In contrast to the triangle analogues, however, **there is no (default) reference quadrilateral**. In each command, you must specify the quadrilateral to be used, either by its four vertices or by the defining polyline.

The quadrilateral objects are less numerous than the triangle objects. OK Geometry can create a large part, but not all, of the objects from vanTienhoven's *Encyclopedia of quadri-objects*.

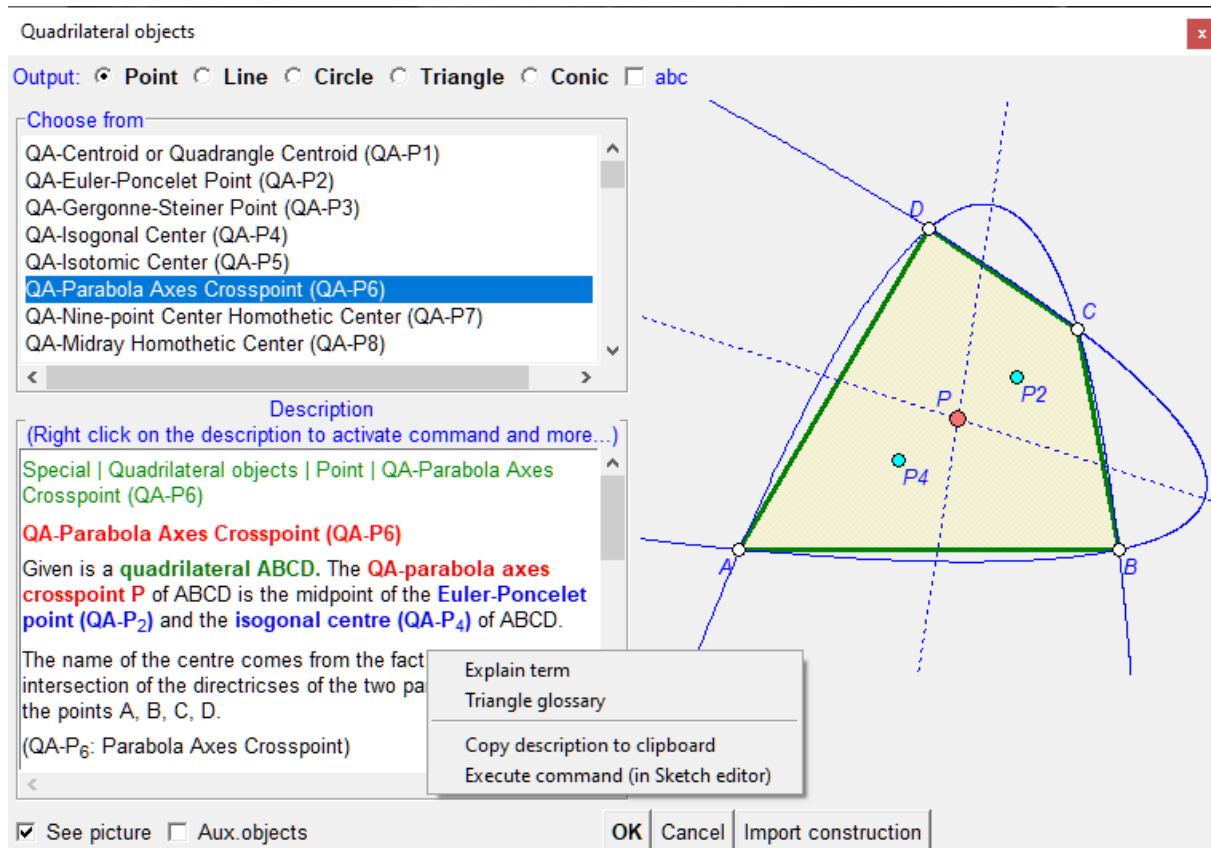


Figure 30

Figure 30 shows a typical group of commands for characteristic objects of a quadrilateral.

- In the first line, select the type of object you want to create (in our case, a point).
- The commands for objects in this group refer to the quadrilateral that you specify after clicking on OK. You can specify the quadrilateral either by its vertices or by its defining polyline.
- As with the triangle objects, a right click on the Description part of the form displays additional commands.

Here is the list of command for quadrilateral objects:

<p><i>A quadrilateral</i></p>	<p>The command creates a quadrilateral in which all four vertices are free.</p> <p><b>Note.</b> The quadrilateral created is not a (default) reference quadrilateral. It also has no cyclic structure attached as is the case of the reference triangle created with the command 'A triangle'.</p> <p><b>Note.</b> The command is active if no objects have been created yet.</p>
-------------------------------	---

<p><i>Quadrilateral objects</i></p>	<p>The quadrilateral objects are organised in several lists:</p> <ul style="list-style-type: none"> <li>- <b>quadrilateral points</b>, e.g., the QA-centroid (QA-P1);</li> <li>- <b>quadrilateral lines</b>, e.g., the QL-Newton line (QL-L1);</li> <li>- <b>quadrilateral circles</b>, e.g., the Miquel circle (QL-Ci3);</li> <li>- <b>quadrilateral triangles</b>, e.g. the QA-diagonal triangle (QA-Tr1);</li> <li>- <b>quadrilateral conics</b>, e.g., the QA-nine-point conic (QA-Co1).</li> </ul>
<p><i>Analyse objects wrt. quadrilateral</i></p>	<p>The command performs a simple observation of the object you click on (point, line, circle, conic). You are immediately asked for the quadrilateral with respect to which the object is to be observed.</p> <p>The output of the command is a list of the quadrilateral's objects that are (in some way) contingent to the observed object.</p> <p><b>Note.</b> Several commands are available to visualise the listed objects and to obtain information about them. You can access them at the bottom of the form or with a right click on the corresponding object on the list of results.</p> <p>The example in Figure 31 shows the results of the observing a line with respect to the quadrilateral ABCD. It is noted that the observed line is parallel to the QL-Steiner line and to the QL-pedal line and that it passes through the QA-Harmonic centre of the quadrilateral ABCD. By a right click on the results or using the commands at the bottom right of the results, we can view the mentioned objects and obtain information about them.</p>

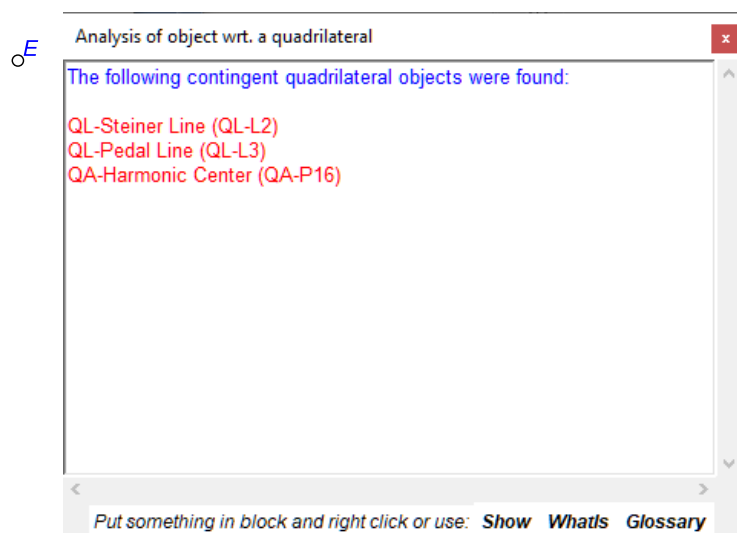
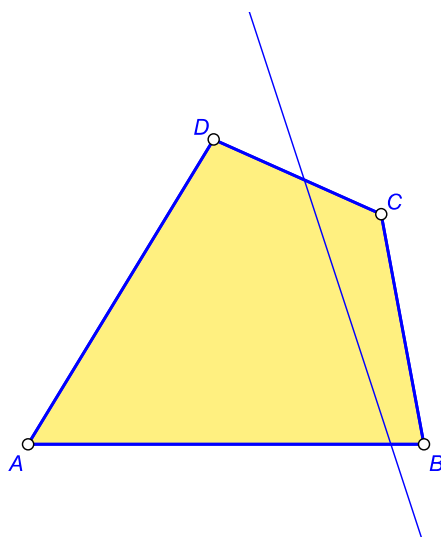




Figure 31

### 3 Advanced query of objects

#### 3.1 Query commands

The observation (**Observe** button) of a configuration provides a list of observed properties. The list is sometimes very long, so it is not easy to focus on the properties that are relevant to us. If you are looking only for properties that involve a specific object, you can use the query buttons (magenta coloured buttons on the right hand side of the displayed construction, e.g.,  ,  , see Figure 32). The simple use of the query commands is explained in the OK Geometry Basic reference manual. In simple terms, the basic query operation filters the properties so that only properties pertaining to specified object are displayed.

When an object is queried, a row of dotted buttons appear above the list of properties (Figure 32, left). The number of dots on the buttons indicates the rigour of the query: the fewer the dots, the less extensive the information displayed, i.e. less relations with less objects are displayed.

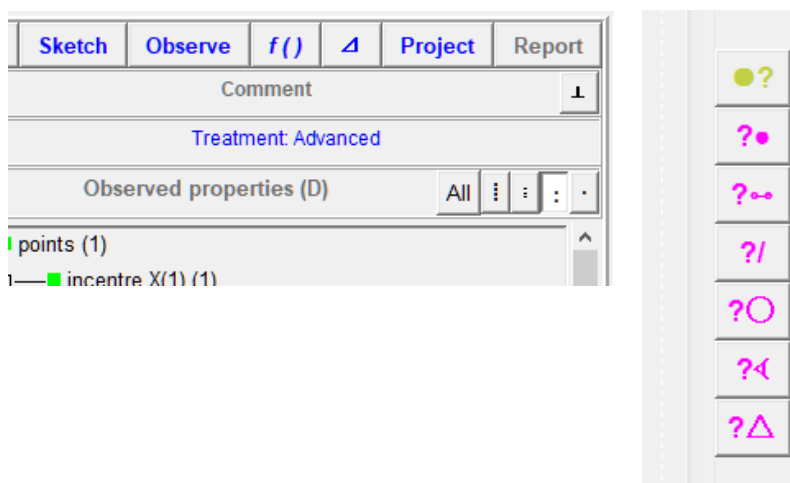



Figure 32

For the sake of completeness, here are the descriptions of the commands associated to the query buttons:

Command	Meaning
<i>Mark/Toggle unknown status</i>	This command (accessible also via button  ) is used to mark/unmark objects that are considered unknown (see Section 3.2). Objects marked as unknown are coloured pale-green. Such objects are not considered in the


	query unless they are the object of the query. A typical use of this function is the study of Euclidean construction problems. In a complete construction, we mark objects that need to be constructed from known objects.
<i>Query labelled point</i>	Click on the point to be queried. In the left pane, only the properties related to the queried point are listed.
<i>Query-Segment</i>	Click on the segment to be queried or its endpoints. In the left pane, only the properties related to the queried segment are listed.
<i>Query-Line</i>	Click on the line to be queried or on two labelled points on it. In the left pane, only the properties related to the queried line are listed.
<i>Query-Circle</i>	Click on the circle to be queried or on three labelled points on it. In the left pane, only the properties related to the queried circle are listed.
<i>Query-Angles</i>	Click on the labelled points (ray, vertex, ray) or on the rays of the angle to be queried. In the left pane, only the properties related to the queried angle are listed.
<i>Query-Triangle</i>	Click on the labelled triangle vertices. In the left pane, only the properties related to the queried triangle are listed.

### 3.2 Known, unknown and auxiliary objects

By default, the query buttons in OK Geometry Plus have an additional functionality called **advanced query**. The advanced query operation of an object attempts to relate that object to a specified subset of objects (called **known objects**) in a configuration and also to many objects (called **auxiliary objects**) that are geometrically derived from the known objects.

The advanced query is of great help in hypothesising solutions to construction tasks. Namely, in construction tasks we start from some **known** objects and try to relate **unknown** objects to the already known ones. The advanced query can also reveal surprising relations between objects in a configuration in other situations.

Constructed objects are **known** by default. You can make an object **unknown** using the command

*Action/Mark/Toggle unknown status*, which is also accessible from the green dot button  on the right side of the screen.

Advanced query is only available in OK Geometry Plus. It is enabled by default, but can be turned off in OK Geometry configuration (*Configure/General options/Observe/Advanced query operations*). Turning the Advanced query OFF will reduce the observation time.

Let us illustrate the use of advanced query with a very simple construction task:

**Example**

Inscribe a square PQRS into a given triangle ABC, as shown in Figure 33 (left)<sup>20</sup>

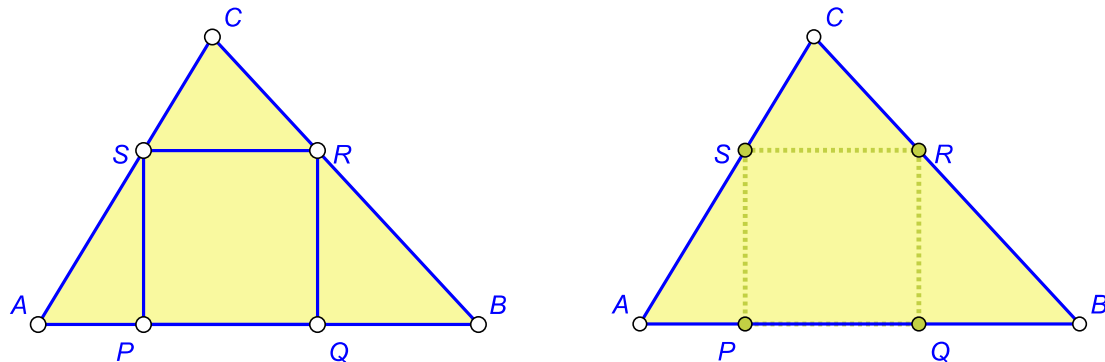




Figure 33

First, we must somehow obtain a final configuration (Figure 33, left). The easiest way is to start with a square PQRS and draw an appropriate triangle ABC around it.

The known objects in our initial construction task are the vertices A,B,C and the three sides of the triangle. The unknown objects are the vertices P,Q,R,S and the four sides of the square.

Using the command **Toggle unknown status** (accessible via the green dot button ) we mark the unknown objects. Unknown points turn pale green, and the unknown lines turn dashed and pale green (Figure 33, right).

At this point we **Observe** the configuration. We obtain the usual list of properties of the configuration. On the far right of the display appear the magenta query buttons for querying specific objects. Advanced query occurs only when we query an unknown object (otherwise a simple query is performed).

We query the point R with the Query point button  (i.e., we first click on the button and then on the point R). After a while, a short list of properties pertaining to the point R appears. At the top of the list you can find new properties that involve auxiliary points derived from known objects. A click on the property visualises this property.

<sup>20</sup> OKExamples\OKG\_Plus\AdvQuery\_01.p

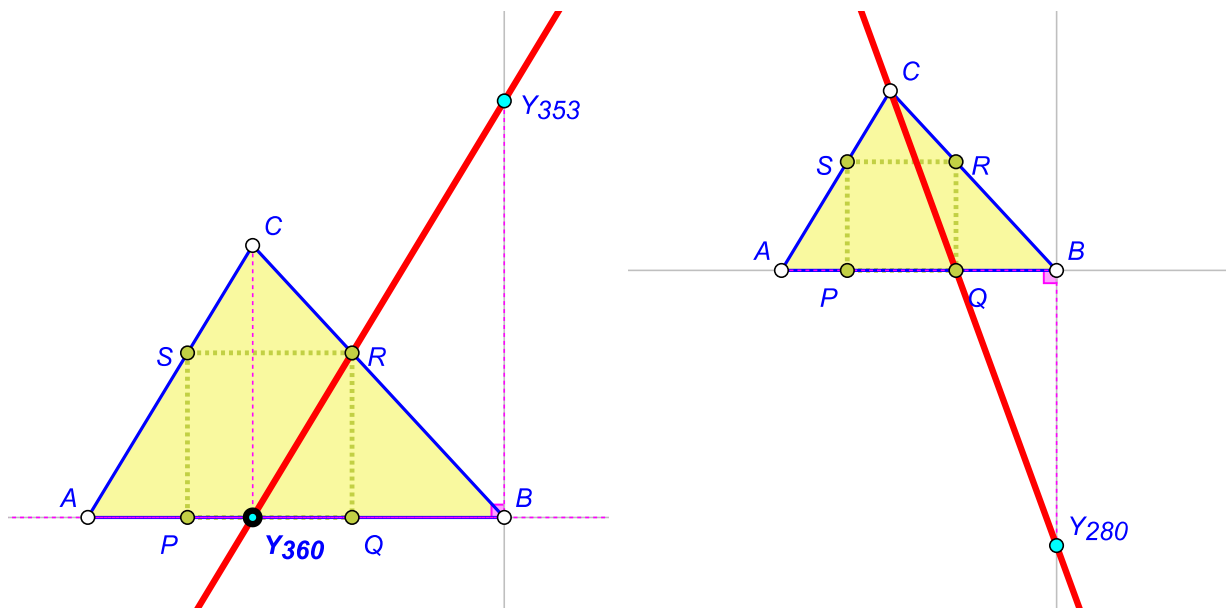


Figure 34

For example: The property *Lines involving R* contains, among other, the relation

*Lines involving R*


R, \_360, \_353

**Note that you can get different names of auxiliary objects for the same relation**, i.e. different numbers with underscore.

Clicking on the property displays the relation (Figure 34, left)<sup>21</sup>.

If you place the cursor on an auxiliary point in the construction, an explanation of the point will appear (you can activate the visual explanation by pressing the **F8** button on the keyboard). In the example shown, \_360 (or Y360) is the orthogonal projection of C onto the line AB, and the point \_353 (or Y353) is point A rotated by 90 degrees clockwise around B.

As usual, right-clicking the construction area opens a context menu with various commands that allow you to export the figure or edit it, include it in a project, etc. In this case the underscores in labels of auxiliary points become Y.

To get information about point Q, apply the **Query point** command  to point Q (no need to **Observe** again). Among the properties you will find one (Figure 34, right) that allows you to construct the point Q (the auxiliary point \_280 is the point A rotated anticlockwise around B by 90 degrees).

Both properties shown are easy to prove.

**Note.** A change in the collection of **unknown** objects requires a new observation with **Observe** command.

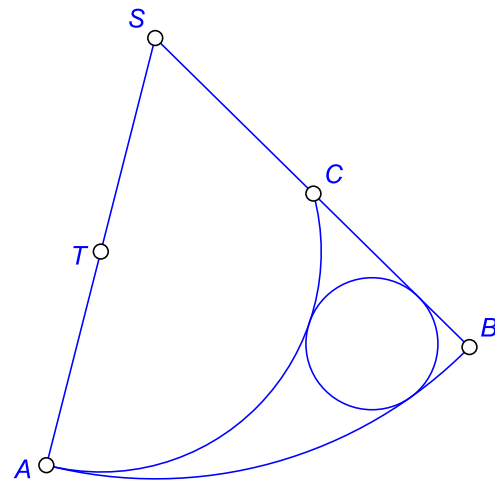
<sup>21</sup> OKExamples\OKG\_Plus\AdvQuery\_02.p

### 3.3 Advanced query examples

#### EXAMPLE 1

In a circle sector with centre  $S$  and  $A, B$  as arc endpoints, there is a second arc. The centre  $T$  of the second arc is the midpoint of  $AS$ , its endpoints are  $A$  and a point  $C$  on the segment  $BS$ .

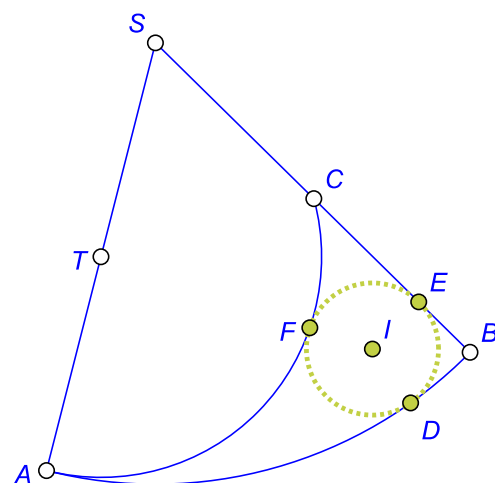
Construct a circle that touches the two arcs and the segment  $BC$ .<sup>22</sup>



When constructing a circle that is tangent to three known objects, it is often advisable to determine the points of tangency of the circle with objects tangent to it.

The inscribed circle is easily drawn with OK Geometry, using the command *Circle 3 objects*. We also draw the points of contact  $D, E, F$  and the centre  $I$  of the circle.

To get clues for a construction, we mark the inscribed circle, the touch points  $D, E, F$ , and the centre  $I$  of the inscribed circle as unknown objects.<sup>23</sup> Then we proceed with the command **Observe**. Then we query the unknown points.



<sup>22</sup> OKExamples\OKG\_Plus\AdvQuery\_03.p

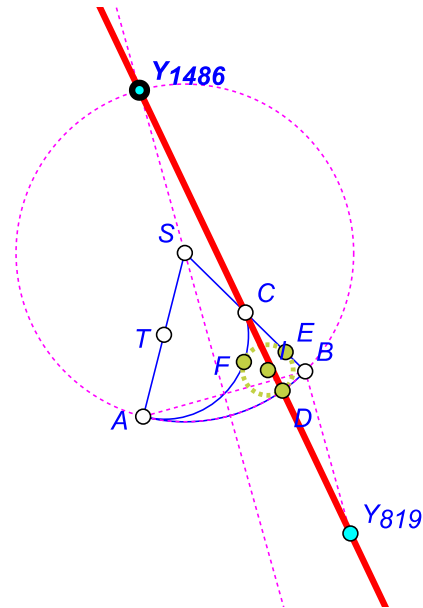
<sup>23</sup> OKExamples\OKG\_Plus\AdvQuery\_04.p



Here is one of several suggestions to obtaining the point D: The point D lays on the line through the auxiliary points Y1486 and Y819, where

- Y1486 is the intersection of the circle  $k(S,A)$  with the bisector of AB;
- Y819 is the image of A rotated by 90 degrees counterclockwise around B.

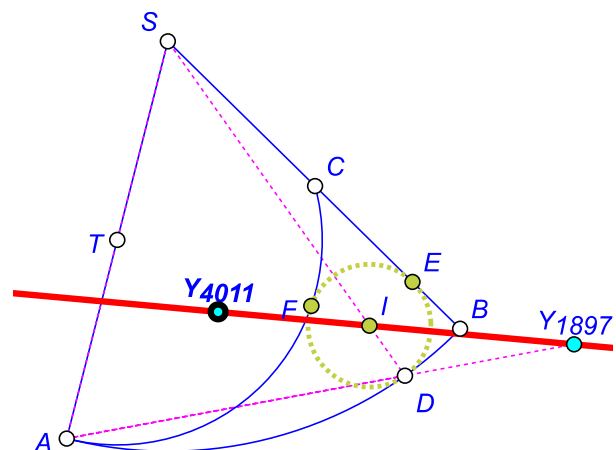
Note that the displayed line does not contain the point C (as it might appear in the figure).



Once the point D is known, it is recommended to mark it as known, and repeat the **Observe** procedure. The advanced query of the centre I gives several suggestions for the construction of I (obviously I lays on the segment SD).

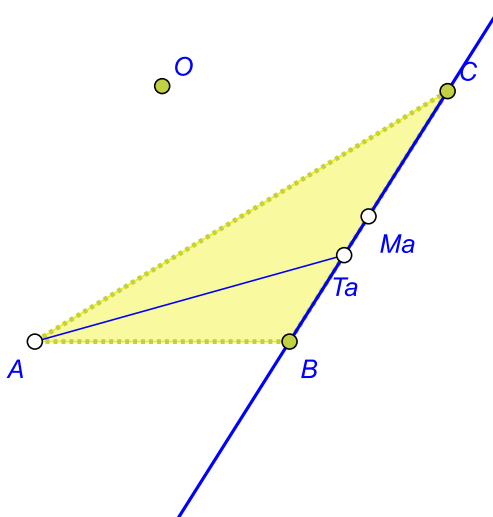
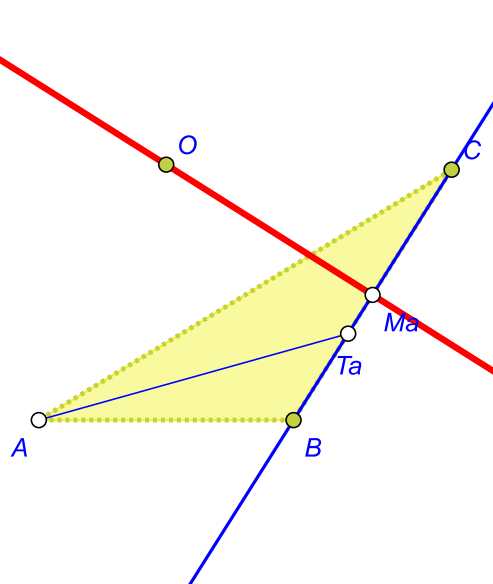
According to one of the suggestions, the point I lays on the line through Y4011 and Y1897, where

- Y4011 is the symmedian point of the triangle ADS, and
- Y1897 is the mirror image of the midpoint of AD in point D.

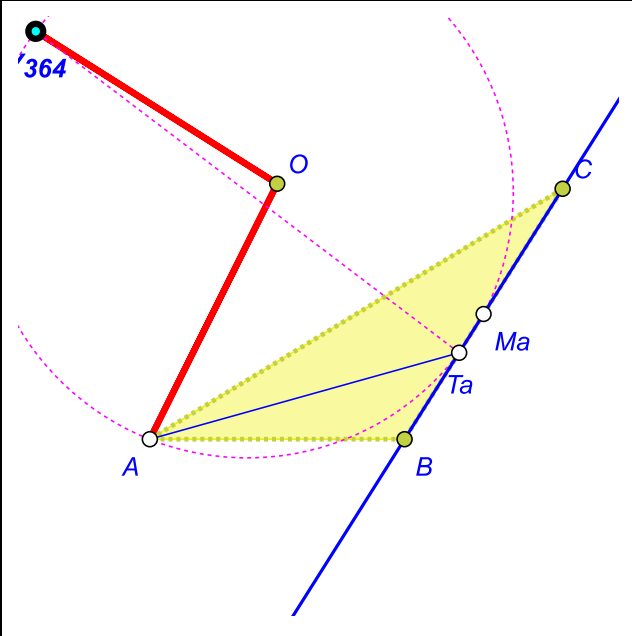
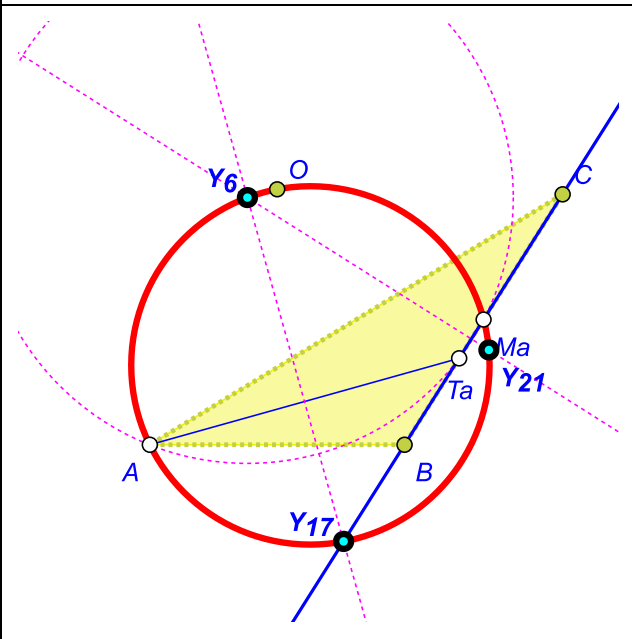


**EXAMPLE 2**

The following example is the 25th problem in the famous Wernick list of construction problems.

	<p>Construct a triangle ABC if the location of the following points of the triangle ABC is known:</p> <ul style="list-style-type: none"> <li>- the vertex A,</li> <li>- the midpoint <math>Ma</math> of the side BC,</li> <li>- the intersection <math>Ta</math> of the angle bisector at A and the side BC.</li> </ul> <p>In a triangle ABC we construct the points <math>Ma</math> and <math>Ta</math>. Note that A, <math>Ma</math>, <math>Ta</math>, and the line BC are known (as is the location of the centroid).<sup>24</sup> We mark everything else as unknown.</p>
	<p>Since observation command and query of points B, C yield nothing useful, we try to determine some other important points of the triangle. The unknown circumcentre O of the triangle ABC is a good choice.</p> <p>An observation and query of the circumcentre O yields several hypotheses that can be quite easily proved. Once we know the position of O, the construction problem will be easily solved.</p> <p>Here is the first (trivial) property: the line <math>O(Ma)</math> is perpendicular to BC in <math>Ma</math>.</p>

<sup>24</sup> OKExamples\OKG\_Plus\AdvQuery\_05.p

	<p>Furthermore, O lays on the bisector of the segment <math>A(Y_{364})</math>, where</p> <ul style="list-style-type: none"> <li>- <math>Y_{364}</math> is the antipode of <math>T_a</math> on the circle through <math>A, T_a, M_a</math>.</li> </ul>
	<p>The circumcentre O also lays on the circle through the points <math>A, M_a, Y_{17}</math>, where</p> <ul style="list-style-type: none"> <li>- <math>Y_{17}</math> is the intersection of the line <math>BC</math> with the bisector of <math>A(T_a)</math>.</li> </ul> <p>Note that O is the antipode of <math>Y_{17}</math> in the mentioned circle, which contains also the points</p> <ul style="list-style-type: none"> <li>- <math>Y_{21}</math> - the intersection of <math>A(T_a)</math> with the bisector of <math>(M_a)(T_a)</math>,</li> <li>- <math>Y_6</math> - the intersection of the line <math>BC</math> with the bisector of <math>(M_a)(T_a)</math>.</li> </ul>

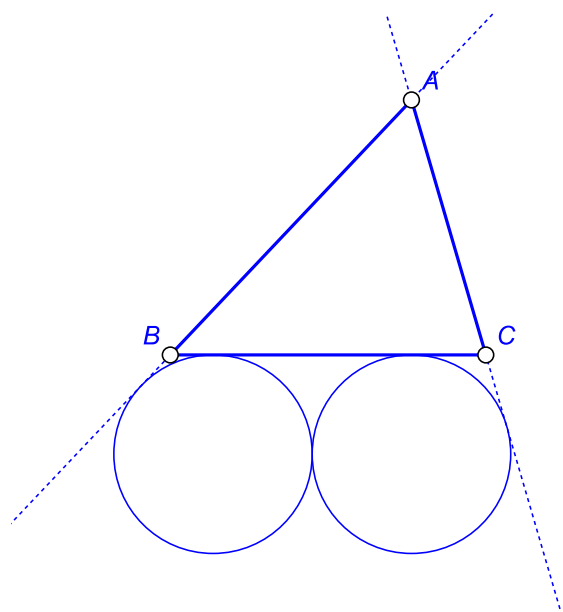
**Example 3**

The following task is related to the Paasche point of triangle (Figure 20).

Given is a triangle ABC. In its exterior we want to construct two congruent circles touching each other externally, both touching the side BC, one touching the line AB and the other touching the line AC.<sup>25</sup>

We are looking for a solution that is nicer than the solution via homothety of a false solution.

To study the construction, we create a described configuration, starting from the congruent circles.

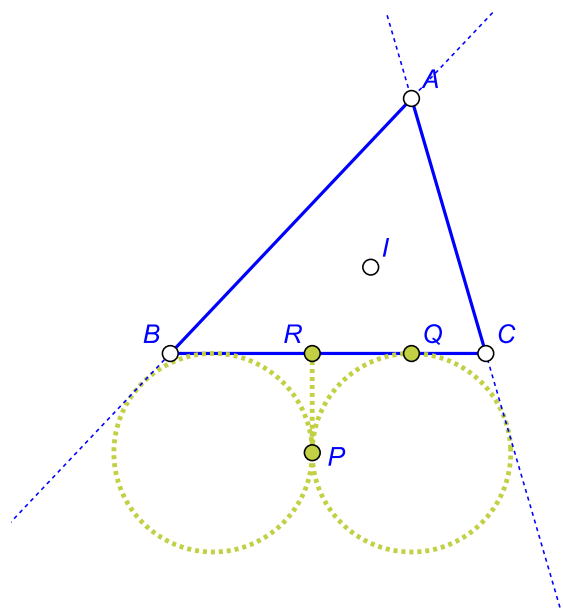


If the vertices of a triangle are known in a construction task, it is sometimes useful to add one or more of its simple centres. A little investigation (and common sense) shows that the incentre I of the triangle ABC is a good choice.

We also add some points as unknown points. If we manage to construct one of them, the task is solved.

We mark the two circles and the added points P, Q, R as unknown.<sup>26</sup>

Then we start the observation.



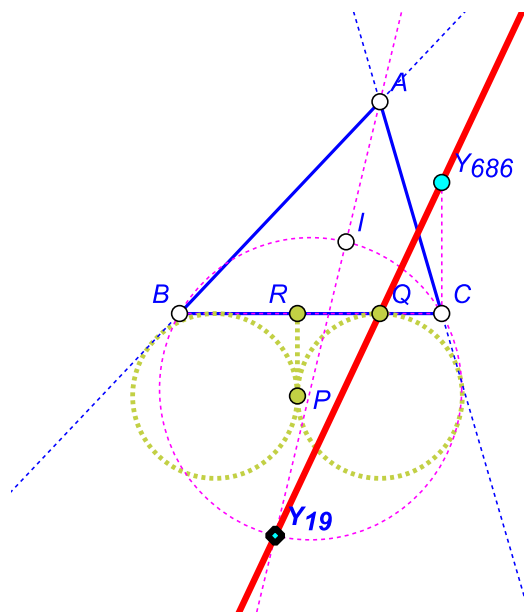
<sup>25</sup> OKExamples\OKG\_Plus\AdvQuery\_06.p

<sup>26</sup> OKExamples\OKG\_Plus\AdvQuery\_07.p

First we query the point Q. OK Geometry suggests that Q is the intersection of the line BC and the line  $Y_{19}Y_{686}$ , where

- $Y_{19}$  is one of the intersections of the circle through B, C, I and the line AI; it is also the antipode of I in this circle;
- $Y_{686}$  is defined by the condition: the angle  $BCY_{686}$  is a right angle;  $|BC| : |CY_{686}| = 2 : 1$ .

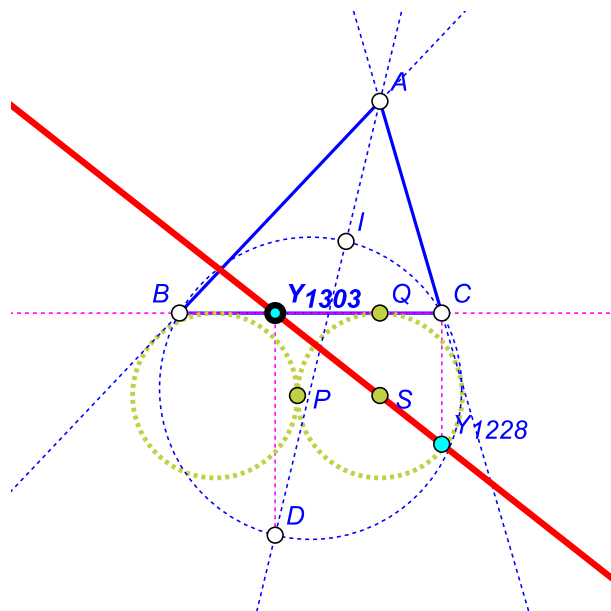
A solution can be found by querying the point R, but not P.



Here is a nice solution for the centre S of one of the two circles. We have added the known point D, the antipode of I.

The point S is the intersection of the line CD (i.e., the angle bisector at C) and the line  $Y_{1228}Y_{1303}$ , where

- $Y_{1228}$  is defined by the condition: the angle  $BCY_{1228}$  is a right angle;  $|BC| : |CY_{1228}| = 2 : 1$ .
- $Y_{1303}$  is the projection of D onto the line BC.



## 4 Observing algebraic relations (formulae)

### 4.1 Introduction

OK Geometry observes geometric properties as well as simple algebraic relations in dynamic constructions. When detecting simple algebraic relations with the **Observe** command in the main menu, OK Geometry considers a whole lot of geometric quantities: all distances between labelled points, all angles between labelled points, and areas of triangles with vertices on labelled points. OK Geometry, for example, detects whether the area of a triangle is the arithmetic mean of the areas of two other triangles in a configuration, or whether some angle between labelled points is twice as large as some other angle in the configuration.

OK Geometry is also capable of observing more complex algebraic relationships between selected geometric quantities in a configuration. This is accomplished with the **Observe formulae** commands that we describe in this section.

The **Observe formulae** command consists of three variants: **simple**, **advanced**, and **triangle**. The variants differ only in the way the geometric quantities are conveyed to the **Observe formulae** command.

- The **simple** variant searches for algebraic relations involving specified quantities, which were measured beforehand in the **Sketch Editor**.
- The **advanced** variant looks for algebraic relations that involve specified geometric quantities that need not to be measured beforehand.
- The **triangle** variant searches for algebraic relations between a specified geometric quantity or ratio of two quantities of a triangle and various unspecified geometric quantities of this triangle.

You can access the *Observe formulae* command via the main menu *Commands/Observe formulae* or via the **f()** button in the main toolbar (Figure 35, left). Then select the desired variant in the form that appears (Figure 35, right). Note that the command is available also in Sketch Editor menu as: *Special/Observe formula*. You can switch between the variants (modes) while using the command.

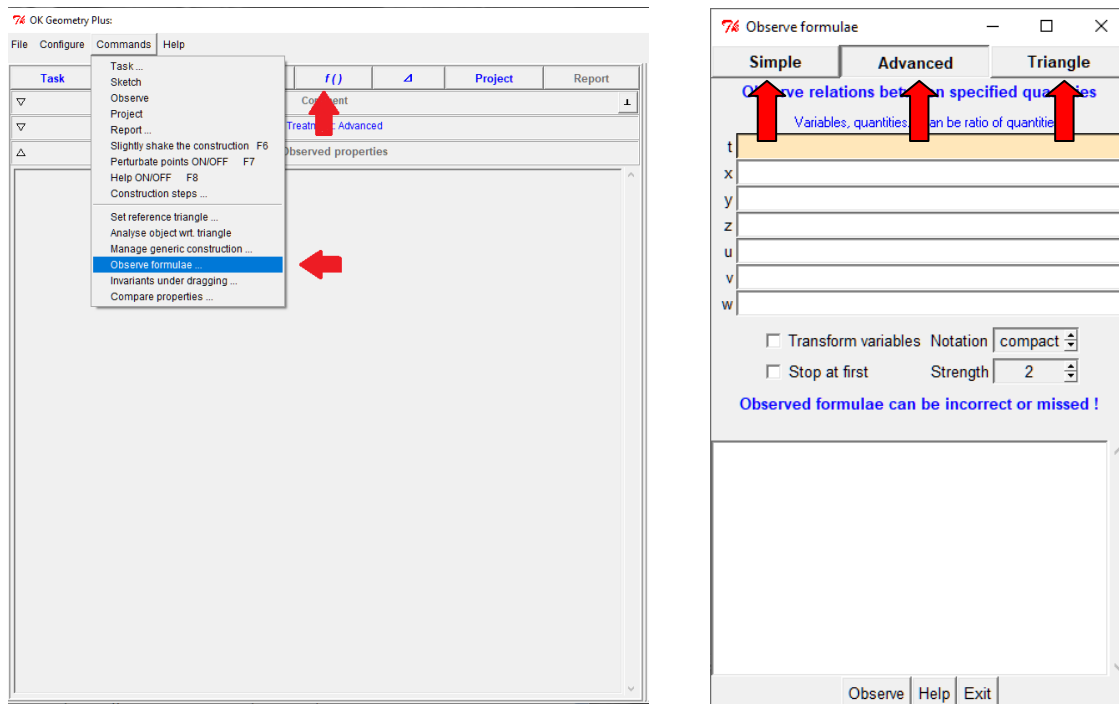


Figure 35

Figure 36 shows a simple illustration of the use of the **Observe formulae** commands. Given is a triangle ABC. Let  $r_i$  be the radius of its incircle. Furthermore, let  $r_a$  be the radius of the circle that touches the sides AB, AC and the incircle and lays between A and the incircle. Define  $r_b$  and  $r_c$  cyclically. We wonder if there is a nice relationship between the radii  $r_a, r_b, r_c, r_i$ .<sup>27</sup>

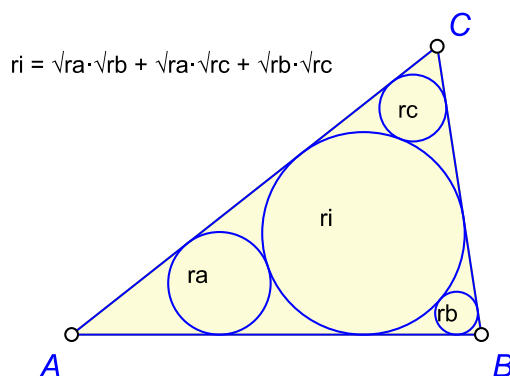


Figure 36

The **simple** variant of the command **Observe formulae** requires that in the dynamic construction we first measure the radii of the four circles. **Observe formulae** then looks for algebraic relationships between the four radii and detects that:

<sup>27</sup> OKExamples\OKG\_Plus\Formula\_01.p

$$ri = \sqrt{ra \cdot rb} + \sqrt{rb \cdot rc} + \sqrt{rc \cdot ra}$$

The **advanced** variant of the command **Observe formulae** performs a similar task, except that the considered quantities do not have to be measured beforehand in the Sketch Editor.

The **triangle** variant of the command **Observe formulae** examines only a single geometric quantity (length or area) or a ratio of two geometric quantities and tries to relate it algebraically to the reference triangle. The triangle variant, for example, detects the following relationships (rewritten by algebraic manipulation) for  $ra$ :

$$ra = r \cdot \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} \cdot \frac{1 - \sin \frac{A}{2}}{1 + \sin \frac{A}{2}}$$

where  $r$  is the circumradius of the triangle. It also detects the following expression for the ratio of  $ra:ri$

$$\frac{ra}{ri} = \frac{1 - \sin \frac{A}{2}}{1 + \sin \frac{A}{2}}$$

It is important to keep in mind that the command **Observe formulae** merely observes whether geometric quantities satisfy some relations that can be (in some way) reduced to polynomial expressions. It does not prove the relations and it may also overlook existing relations. Thus, it is a useful heuristic tool for obtaining hypotheses that should be proved in a mathematically decent way. To prove and to geometrically interpret the mentioned algebraic observations related to Figure 36, for example, is a nice recreational task.

The process of observing formulae may take time, especially for the **triangle** variant. For this reason you can monitor the fraction of work done and you can also interrupt the observation process.

In the following sections we explain how to use the three variants of the **Observe formulae** command.

## 4.2 Observe formulae – simple

In this section we show how to use of the **simple** variant of the command.

We first explain the procedure on a very trivial example.

### Example

How does the area of an isosceles trapezium relate to its sides  $a, b, c, d$ ? <sup>28</sup>

1. Provide a dynamic construction of the object(s) under consideration. The construction can be imported from some dynamic geometry software or created in OK Geometry Sketch Editor.
2. Measure the potentially relevant geometric quantities in the Sketch Editor. You can use the commands *Number/Distance*, *Number/Radius*, *Number/Length* or *Number/Area*. In our case we have measured the sides  $a, b, c, d$  and the area  $Ar$  (Figure 37).

<sup>28</sup> OKExamples\OKG\_Plus\Formula\_02.p



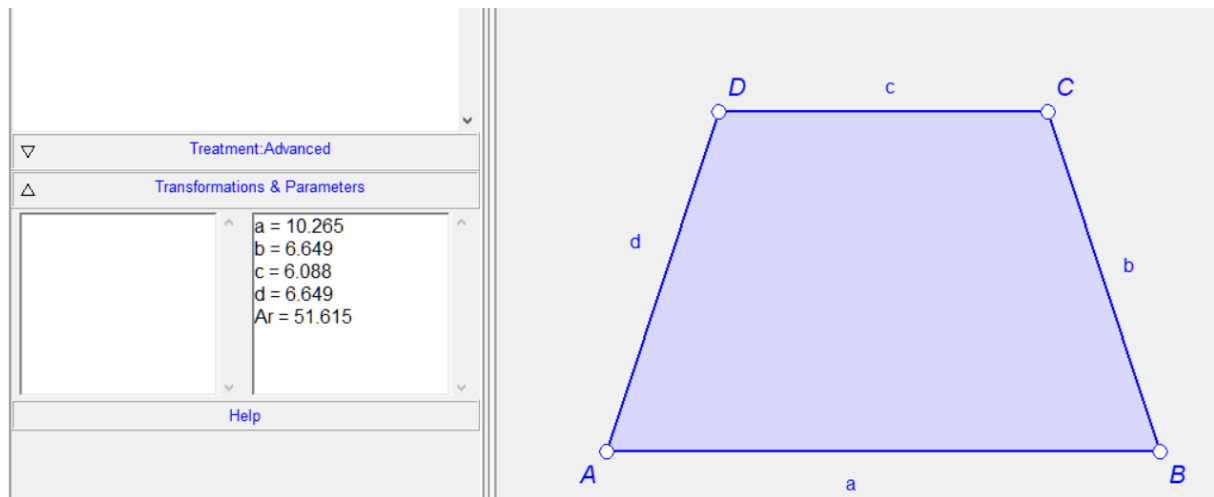


Figure 37

3. Apply the command *Command/Observe formulae/Simple*. A form appears. In the list of measured quantities, select the ones you think are relevant. In our case we have selected all of them. Then click the **Observe** button (Figure 37, left).

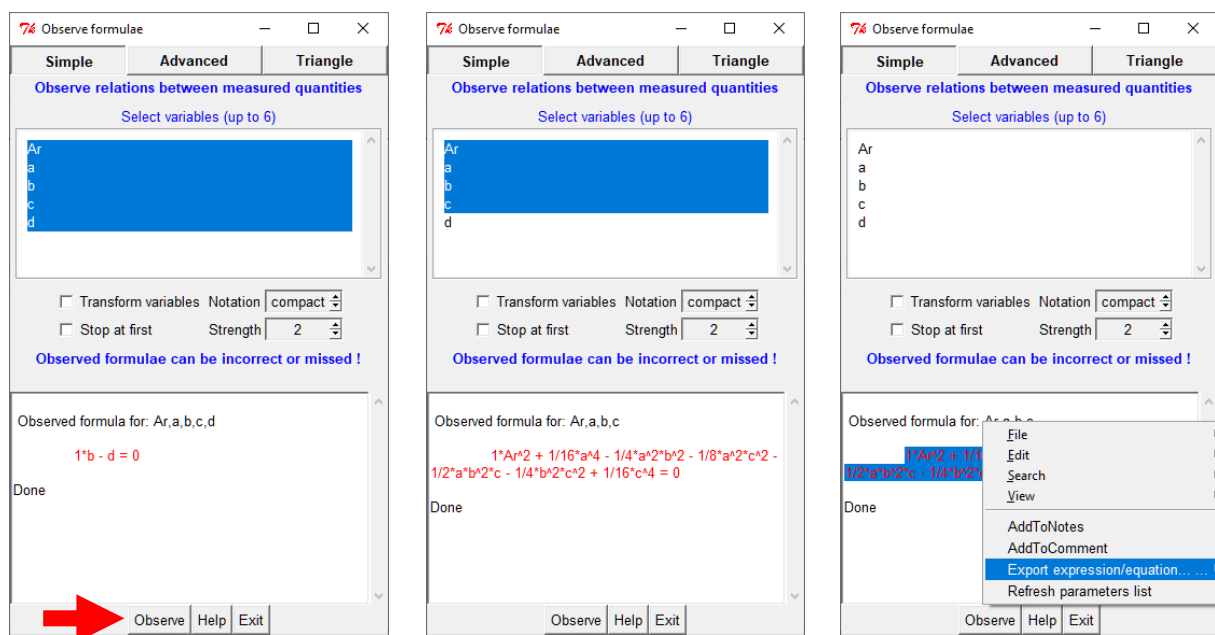


Figure 38

4. OK Geometry looks for a polynomial relation of the lowest degree containing the selected variables. In our case there is a trivial relation  $b = d$ , which is certainly not of our interest. Therefore we unselect a redundant variable  $b$  or  $d$  and repeat the observation (Figure 38, centre).
5. The obtained polynomial relation usually requires additional algebraic manipulation, depending on your aesthetic standards and mathematical goals.
6. To add a further dynamic numerical test for the obtained relation, proceed as follows (Figure 38, right):
  - a. Put the considered algebraic relation in a block.

- b. Right click on the relation and select the command *Export expression/Add as parameter in construction*.
- c. A parameter *Fcondx* is added to the dynamic construction. The parameter numerically checks the correctness of the equation in the block. You can further dynamically modify the construction and observe if *Fcondx* has the value 'True'. Note that you can declare as a parameter only a part of the relation by putting in block only this part of the relation.

#### 4.2.1 Comments on observed formulae

It is important to be aware of the status of the relations found regardless of the variant used (**simple**, **advanced** or **triangle**).

1. **Observing formulae** command considers specified geometric quantities of a dynamic construction in order to detect polynomial relationships between them or relations that can be reduced to a polynomial relationship of degree up to 4. If several variables are involved, the maximal degree can be 3.
2. If one or more relations of a certain degree are found, **Observe formulae** will not search for relations of higher degree.
3. **The observed algebraic relations are the result of numerical observations of the dynamic construction. The relations found are not proven and there is no guarantee they are correct. A relationship can also be overlooked.** In general, overlooking a relation occurs more likely than finding a relation that is not valid at least locally.

#### 4.2.2 Setting the parameters for observing formulae

**Observing formulae** commands have a few options:

<b>Notation</b>	<p>The <b>compact</b> option uses some idiosyncratic notation that is explained in the advanced and the triangle mode of Observe formulae. For example, sA2 stands for <math>\sin(A/2)</math>.</p> <p>The <b>extended</b> option, when necessary, adds to compact formulae a readable version of them.</p> <p>The <b>xyz</b> option writes the formulae in terms of the variables <math>t, x, y, z, u, v, w</math>. This way the expressions are easier to read and can be more easily recognised by computer algebra systems.</p>
<b>Transform variables</b>	<p>If checkmarked, OK Geometry will look for polynomial relations that involve selected variables and also transformed variables (e.g. the squares of variables). In this case the calculation takes more time.</p>
<b>Stop at first</b>	<p>In many cases <b>Observe formulae</b> searches for several results. For example, if 'Transform variables' is checkmarked, results are searched for each type of transformation of variables. The <b>triangle</b> variant of <b>Observing formulae</b> considers even more types of results, and all this can be time-consuming.</p> <p>In order to reduce the calculation time, checkmark the 'Stop at first' option. In this</p>

	case the computation will stop as soon as the first relation is found.
<b>Strength</b>	<p>A higher strength means a higher probability of finding a relation (at the cost of longer computation time). You can set the strength from 1 to 4, the default value is 2.</p> <p>The usual strategy is to work with strength 2. If the observation does not find a relationship then try again with a higher strength.</p>

### 4.2.3 Managing the results of Observe formulae

The results of the command **Observe formulae** are written in an ordinary text editor. In addition to the common text editor commands, available by a right-clicking on the results area, there are some specific commands (see Figure 38, right). All specific commands refer to the text in block. **So, to use the specific commands, you must first put in block the text on which to apply the command.**

Here is a brief description of the commands:

**Add to Notes.** The text in the block will be added to the Notes section of the project report.

**Add to Comment.** The text in the block will be added to the Comment section of the current task.

**Export expression/equation.** A **single relation** or **part of a single relation** in **compact** or **extended notation** can be put in a block and exported in different ways for different purposes.

- **As triangle expression.** The relation in the block is converted so that it can be used in the command *Number/Triangle expression* of the Sketch Editor.
- **As Xcas/Giac input.** The relation in block is converted so that it can be used as input for algebraic manipulation in Xcas/Giac software.
- **As Derive input.** The relation in block is converted so that it can be used as input for algebraic manipulation in Derive software. The Derive input options should be set to: Input mode = Word, Case sensitivity = Sensitive.
- **As Maxima input.** The relation in block is converted so that it can be used as input for algebraic manipulation in Maxima/wxMaxima software.
- **As Mathematica input.** The relation in block is converted so that it can be used as input for algebraic manipulation in Mathematica software.
- **As parameter in construction.** This operation can be used as a check of the relation under a manual dragging of the dynamic construction. The expression in the block is converted to a parameter of the current construction. If the block contains the = symbol, the generated parameter is a condition *Fcondx*, otherwise it is an ordinary numeric parameter called *Fexprx*.

**Refresh parameter list.** Use this command when, while observing formulae in the **Simple mode** you define new parameters as geometric quantities. So, during the process of formulae observation you can define new parameters as geometric quantities in the Sketch Editor and refresh the parameter list.

You may find useful also the ordinary **Copy** command for transferring **single relations** or **part of relations** to computer algebra systems (CAS). The xyz notation is probably the best option for this operation.

#### 4.2.4 More examples

##### Example 1

Given is a triangle ABC. Let  $A'$ ,  $B'$ ,  $C'$  be the points of contact of the incircle of ABC with the sides BC, CA, AB of the triangle ABC (Figure 39). Let  $ra, rb, rc, rr$  be the radii of respective incircles of the triangles  $AC'B'$ ,  $BA'C'$ ,  $CB'A'$ ,  $A'B'C'$ .<sup>29</sup>

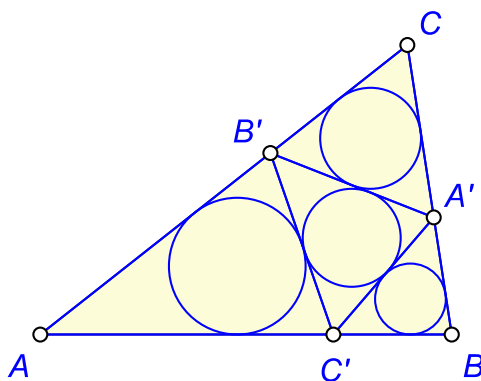


Figure 39

The **Observe formulae** command detects the relationship

$$4 \cdot ra \cdot rb \cdot rc = (ra + rb + rc + rr) \cdot rr^2$$

Can you prove this relation?

##### Example 2

Given a triangle ABC, let  $rA, rB, rC$  be the radii of the mutually externally tangent circles with centres A, B, C. Furthermore, let  $rI$  and  $rO$  be the radii of the circles externally and internally tangent to the three mentioned circles (i.e. the inner and outer Soddy circles). (Figure 40)<sup>30</sup>

<sup>29</sup> OKExamples\OKG\_Plus\Formula\_03.p

<sup>30</sup> OKExamples\OKG\_Plus\Formula\_04.p

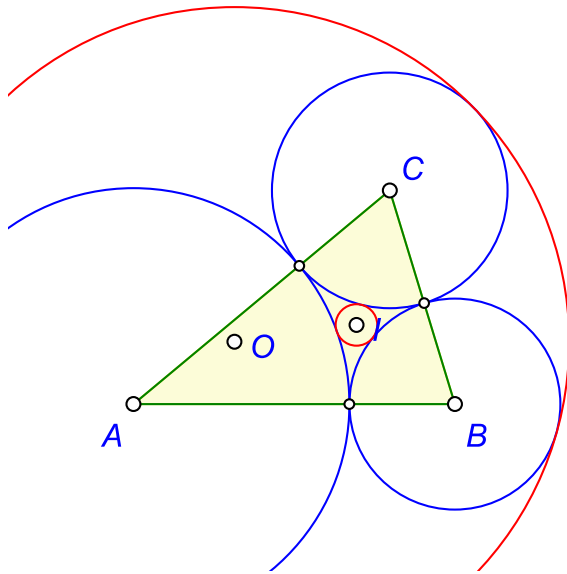


Figure 40

**Observe formulae** suggests nice relationships between variables

- $r_A, r_B, r_C, r_I$
- $r_A, r_B, r_C, r_O$
- $r_A, r_B, r_C, r_I, r_O$

Note. Checkmark the **Transform variables** option.

### Example 3

Let  $H$  be the orthocentre of the acute triangle  $ABC$ . Let  $r_a, r_b, r_c$  be the radii of incircles of the triangles  $HBC$ ,  $HCA$ , and  $HAB$  respectively. Furthermore, let  $r_i$  and  $r$  be the radii of the incircle and the circumcircle of the triangle  $ABC$ . (See Figure 41.)<sup>31</sup>

<sup>31</sup> OKExamples\OKG\_Plus\Formula\_05.p

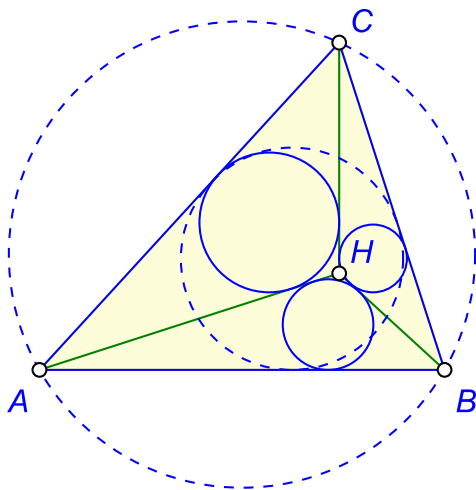


Figure 41

**Observe formulae** suggest this relationship between the 5 radii:

$$\begin{aligned}
 4 * (ra^2 + rb^2 + rc^2 + r^2) &= \\
 &= (ra + rb + rc + ri - 2r)^2 + 8 \cdot r \cdot ri
 \end{aligned}$$

Does the relation also hold for obtuse triangles?

### 4.3 Observe formulae – advanced

The **advanced** variant of **Observe formulae** is similar to the **simple** one, except that you can also directly specify the involved geometric quantities of the dynamic construction. Thus, the geometric quantities to be considered in a relation do not have to be measured beforehand in the Sketch Editor.

In this section we will show how to use of the **advanced** variant of the command.

The first entry of the form that appears (Figure 43) is lightly coloured to emphasise its special role:

- In the first entry (t) you can write a geometric quantity or a ratio of two quantities (of the same dimension). Note that the remaining entries can only contain geometric quantities (or are empty).
- If the first entry is not empty, OK Geometry searches only for formulae that contain this entry.

We explain the **advanced** variant of the command with a simple example:

In a convex quadrilateral ABCD let E, F, and G be the midpoints of sides BC, CD, and DA. Furthermore, let P and Q be the intersection of line GE with the lines AF and BF (Figure 42, left).

How are related the lengths of segments AP, PF, BQ, and BF? (See Figure 42, left.)<sup>32</sup>

<sup>32</sup> OKExamples\OKG\_Plus\Formula\_06.p

What is the relationship between the areas of triangles labelled as  $t$ ,  $x$ ,  $y$ ,  $z$ ? (See Figure 42, right.)<sup>33</sup>

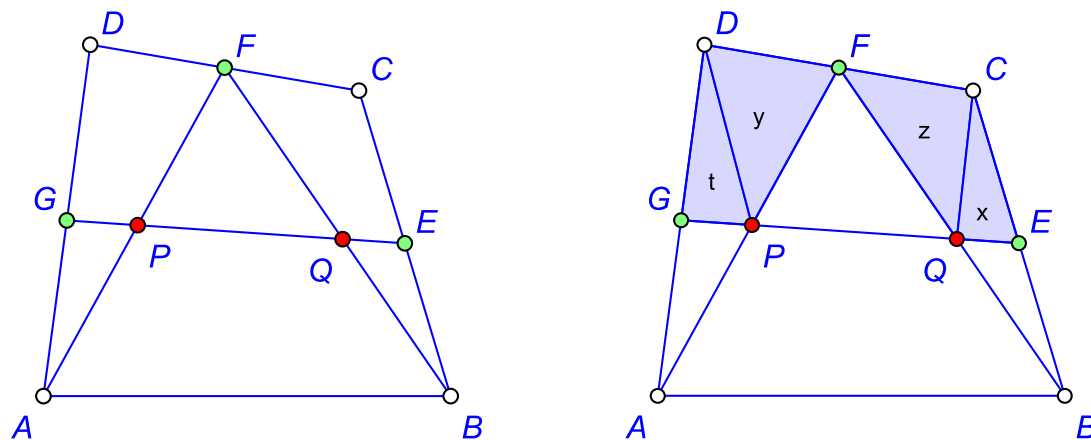


Figure 42

Proceed as follows:

1. Provide a dynamic construction of the object(s) under consideration. The construction can be imported from dynamic geometry software or created in the OK Geometry Sketch Editor.
2. Apply the command *Command/Observe formulae/Advanced*. A form appears. In the entries  $t$ ,  $x$ , ... write the quantities to be considered. In our case we write  $d(AP)$  for the length of the segment  $AP$  and similarly for the other three segments. Then click on the **Observe** button (Figure 43, left).
3. The polynomial relation obtained usually requires additional algebraic manipulation, depending on your aesthetic standards and mathematical goals. The found relation can be rewritten as

$$\frac{|AP|}{|PF|} + \frac{|BQ|}{|QF|} = 2$$

Remember that we are dealing with a result of an observation. It is a nice exercise in geometry to actually prove the correctness of the formula.

4. To add a further dynamic numerical test for the obtained relation proceed as follows:
  - a. Insert the considered relation into a block. The expression must be in **compact** or **extended notation**.
  - b. Right click on the relation and select the command *Export expression/equation/as parameter in construction*.
  - c. A parameter  $Fcondx$  is added to the dynamic construction. The parameter numerically checks the correctness of the equation in the block. You can dynamically modify the construction and observe if  $Fcondx$  has the value 'True'. In our case, we can dynamically check if convexity of the quadrilateral  $ABCD$  is a necessary condition

<sup>33</sup> OKExamples\OKG\_Plus\Formula\_07.p

for the relation to be true. Note that you can as well declare as a parameter only a part of the relation by putting only that part of the relation in a block.

- The problem with areas is handled in a similar way. In order to consider the area of triangle PGD we fill the entry t as Area(PGD). Note that the result of the observation is written differently because we have chosen the notation option 'xyz'.

$$\text{Variables: } t=\text{Area(PGD)} , x=\text{Area(CEQ)} , y=\text{Area(PFD)} , z=\text{Area(CFQ)}$$

$$1*t*z + x*y - y*z = 0$$

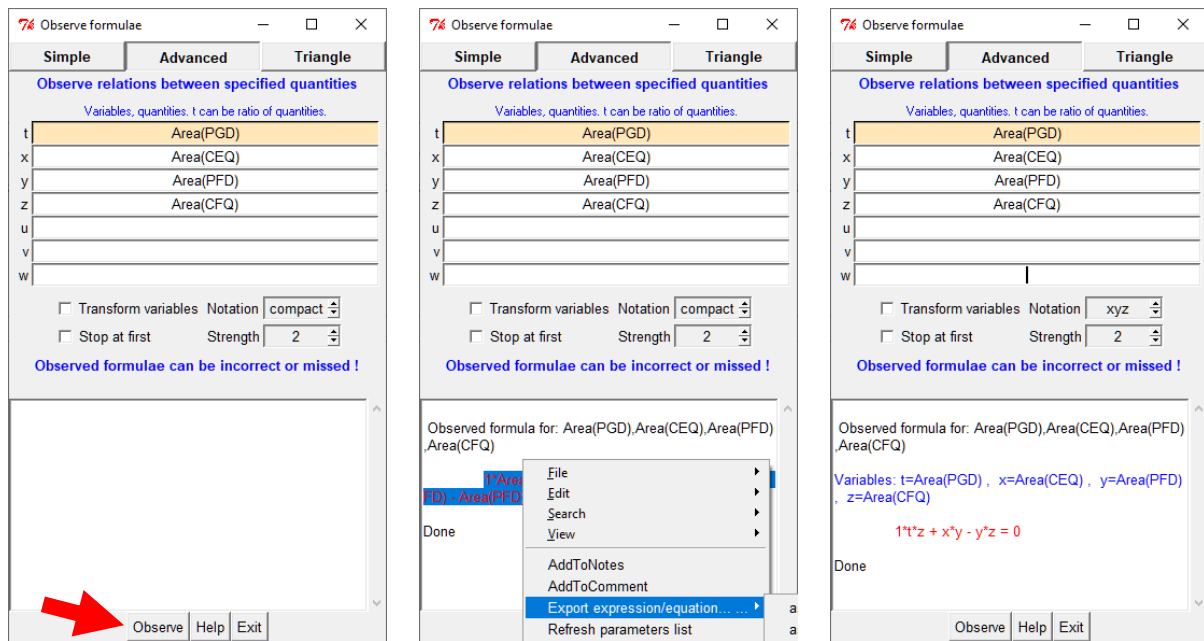


Figure 43

#### 4.3.1 Comments on observed formulae

Many problems can be solved with the above scheme, i.e. by considering the distance between the points P and Q as  $d(PQ)$  and the area of the triangle PQR as  $\text{Area}(PQR)$ . However, the **advanced** variant can handle more complex situations. Here are the conventions to be followed in describing the geometric quantities in the entries t, x, y, ...:

- The entries can also be quantities measured in the Sketch Editor (as parameters) with the measuring commands (Distance to object, Length, Perimeter, Area). The parameters from the Sketch Editor can be written in brackets, e.g.,  $[par]$ , or without brackets, e.g.,  $par$ .
- The entries can also be parameters obtained by computation in the Sketch Editor. In this case **if the parameter is 1-dimensional, its name should start with a lowercase letter, and if the parameter is 2-dimensional quantity, its name should start with an uppercase letter.**
- The first entry (t) may contain a geometric quantity, e.g.  $d(AB)$ , or **a ratio of two quantities** of the same dimension, e.g.  $d(AB):d(CD)$  or  $\text{Area}(ABP):\text{Area}(CAP)$ .
- In addition to  $d(PQ)$  and  $\text{Area}(PQR)$  you can use a whole range of geometric quantities that are specified within the entry. The list of all allowed geometric quantities is in Section 4.3.2.



- It is possible to fill an entry with more than one quantity, separated by a space. In this case, each of the space-separated quantity is considered separately.
- The options of the command are the same as described in Section 4.2.2. The treatment of the results is the same as described in Section 4.2.3.

#### 4.3.2 Symbolising geometric quantities

Two symbols are used for distance:

$d(AB)$	distance between the points A and B
$o(ABC)$	oriented distance from the point A to the oriented line BC

The quantities listed below refer to triangles. **If the triangle argument is missing, the quantity refers to the reference triangle.** For example:

$\text{Area}(PQR)$  ..... denotes the area of the triangle PQR,

$\text{Area}$  ..... denotes the area of the reference triangle (if it exists),

$a(PQR)$  ..... denoted the length of the side QR in triangle PQR,

$a$  ..... denotes the length of the side  $a$  of the reference triangle (if it exists).

$a(ABC)$	length of the side BC
$b(ABC)$	length of the side CA
$c(ABC)$	length of the side AB
$s(ABC)$	half-perimeter of the triangle ABC
$r(ABC)$	radius of the circumcircle of the triangle ABC
$r_i(ABC)$	radius of the incircle of the triangle ABC
$\text{Area}(ABC)$	Area of the triangle ABC
$S(ABC)$	$2 \cdot \text{Area}(ABC)$
$SA(ABC)$ $SB(ABC)$ $SC(ABC)$ $SW(ABC)$	Conway parameters for triangle ABC

The following quantities also refer to triangles. Each quantity in the list consists of three cases related to the three angles of the triangle argument. We list only the cases related to the first angle of the triangle. If the triangle argument is missing the quantity refers to the reference triangle (if it exists).

For example:

$$cA(PQR) = r \cdot \cos(P),$$

$$cB(PQR) = r \cdot \cos(Q),$$

$$cC(PQR) = r \cdot \cos(R), \text{ where } r \text{ is the circumradius of } PQR.$$

$cA(ABC)$	$r \cdot \cos(A)$	$r$ .... circuradius of the triangle ABC $A, B, C$ .... angles of the triangle ABC $W$ .... Brocard angle of the triangle ABC
$tA(ABC)$	$r \cdot \tan(A)$	
$cA2(ABC)$	$r \cdot \cos(A/2)$	
$sA2(ABC)$	$r \cdot \sin(A/2)$	
$tA2(ABC)$	$r \cdot \tan(A/2)$	
$cA3(ABC)$	$r \cdot \cos(A/3)$	
$sA3(ABC)$	$r \cdot \sin(A/3)$	
$tA3(ABC)$	$r \cdot \tan(A/3)$	
$cW(ABC)$ $sW(ABC)$ $tW(ABC)$	$r \cdot \cos(W)$ $r \cdot \sin(W)$ $r \cdot \tan(W)$	

### 4.3.3 More examples

#### Example 1

Let  $G$  be the centroid of the triangle  $ABC$ . Let  $A'$  be the intersection, other than  $A$ , of the line  $AG$  with the circumcircle of  $ABC$ . Define  $B'$  and  $C'$  cyclically (Figure 44)<sup>34</sup>.

Consider the following areas:

$$t = \text{Area}(ABC)$$

$$x = \text{Area}(A'BC)$$

$$y = \text{Area}(AB'C)$$

$$z = \text{Area}(ABC')$$

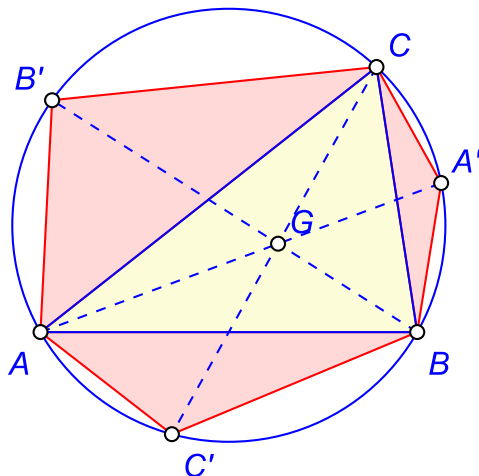


Figure 44

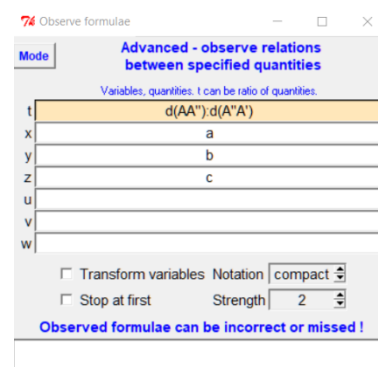
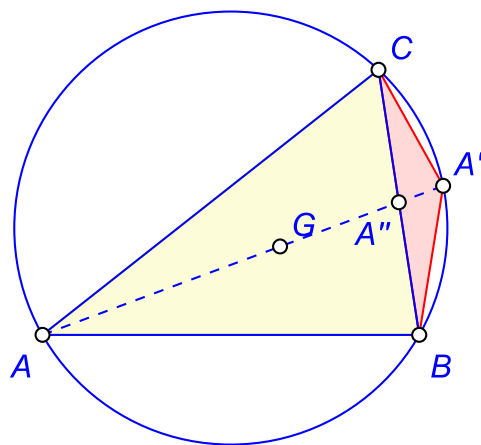
<sup>34</sup> OKExamples\OKG\_Plus\Formula\_08.p

The advanced variant of **Observe formulae** command suggests that the following (reordered) relation holds:

$$\left(\frac{x}{t} + \frac{y}{t} + \frac{z}{t}\right) + 3 \cdot \frac{x}{t} \cdot \frac{y}{t} \cdot \frac{z}{t} \cdot \left(\frac{t}{x} + \frac{t}{y} + \frac{t}{z} + 9\right) = 1$$

### Example 2

To shed more light on the configuration in Figure 44, we consider the ratio  $|AA''| : |A''A'|$  in the configuration on Figure 45 (left). We try to express this ratio in terms of the sides of the triangle ABC. Note that if ABC is declared as a reference triangle we can use the symbols  $a, b, c$  for the length of the sides of the triangle ABC (Figure 45, right)<sup>35</sup>.



$$\frac{d(AA'')}{d(A''A')} = \frac{-a^2 + 2b^2 + 2c^2}{a^2}$$

Figure 45

## 4.4 Observing formulae - triangle

The **triangle** variant of **Observe formulae** is a heuristic tool that suggests how to express a geometric quantity or a ratio of two geometric quantities in terms of some 'commonly used triangle quantities' such as length of its sides. This variant is quite easy to use, unfortunately it is sometimes time consuming. We explain the use of the triangle variant with a trivial example.

### Example

Given a triangle ABC, let  $A'$  be the centre of its A-excircle, i.e.  $A'$  is the centre of the circle that lays outside the triangle ABC and touches the extension of the side AB, the extension of the side AC, and

<sup>35</sup> OKExamples\OKG\_Plus\Formula\_09.p

the side BC. Furthermore, let  $A''$  be the point of contact of the A-excircle with the side BC (Figure 46).<sup>36</sup>

We want to express somehow the radius of the A-excircle, i.e.  $|A'A''|$ , as well as the ratio  $|BA''|:|A''C|$ .

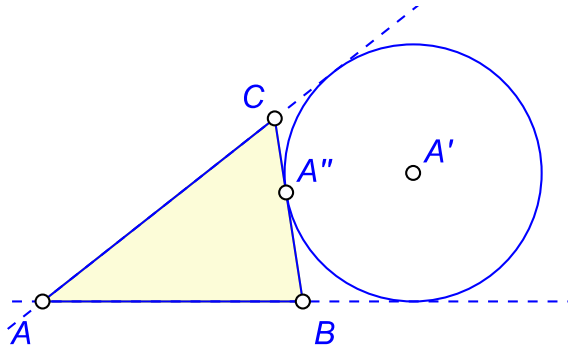


Figure 46

We proceed as follows:

- Provide a dynamic construction of the object(s) under consideration. The construction can be imported from dynamic geometry system or created in the OK Geometry Sketch Editor.  
**Declare the triangle ABC as the reference triangle.**
- Activate the command *Command/Observe formulae/Triangle*. A form will appear. In the first entry write the considered (Figure 47, left) quantity or the ratio of two quantities (Figure 47, centre): in our case,  $d(A'A'')$  for the first observation, and  $d(BA''):d(A''C)$  for the second observation. Then click on the **Observe** button.

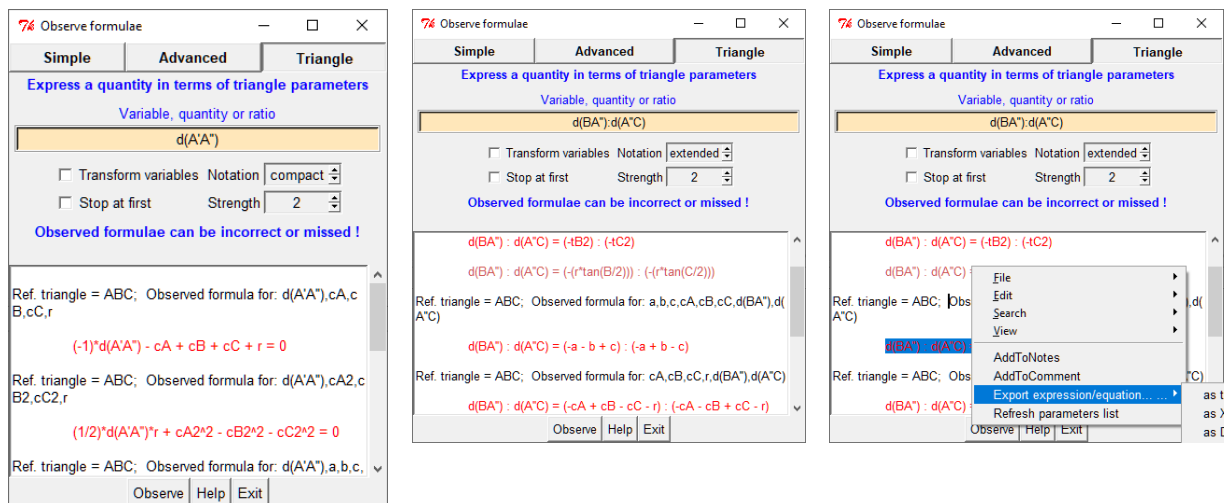


Figure 47

- OK Geometry looks for various ways of expressing the considered quantity or ratio in terms common triangle elements. Some of the suggestions require an interpretation as explained

<sup>36</sup> OKExamples\OKG\_Plus\Formula\_10.p

in Section 4.3.2. The highlighted part of the results in Figure 47 (left) and Figure 47 (right), for example, can be rewritten as

$$\begin{aligned}\frac{d(A'A'')}{2} - r \cdot \cos\left(\frac{A}{2}\right)^2 + r \cdot \cos\left(\frac{B}{2}\right)^2 + r \cdot \cos\left(\frac{C}{2}\right)^2 &= 0 \\ d(A'A'') \cdot ri &= (a - b + c)(a + b - c) \\ \frac{d(BA'')}{d(A''C)} &= \frac{\tan \frac{B}{2}}{\tan \frac{C}{2}} \\ \frac{d(BA'')}{d(A''C)} &= \frac{a + b - c}{a - b + c} \\ \frac{d(BA'')}{d(A''C)} &= \frac{\cos A - \cos B + \cos C + 1}{\cos A + \cos B - \cos C + 1}\end{aligned}$$

In the above expressions, according to Section 4.3.2,  $r$  is the radius of the circumcircle of  $ABC$ ,  $ri$  is the radius of the inscribed circle,  $a$ ,  $b$ ,  $c$  are the side lengths, and  $A$ ,  $B$ ,  $C$  are the angles of the triangle  $ABC$ .

We emphasize again that the **Observe formulae** command does not provide a proof of the detected relations. In the above case one can easily provide geometric proofs for the proposed relations.

In general, the list of results can contain several items or can be empty.

- To add a further dynamic numerical test for the obtained relation proceed as follows:
  - a. Insert the considered relation into a block. The expression must be in **compact** or **extended notation**.
  - b. Right click on the relation and select the command *Export expression/equation|as parameter in construction*.
  - c. A parameter  $Fcondx$  is added to the dynamic construction. The parameter numerically checks the correctness of the equation in the block. You can modify the construction dynamically and observe if  $Fcondx$  has the value 'True'. Note that you can declare only a part of the relation as parameter by putting in block only this part of the relation.

#### 4.4.1 Comments on observed formulae

- **Observed algebraic relations are a result of numerical observations of the dynamic construction. The found relations are not proved and there is no guarantee they are correct. A relation can also be missed. In general, a missed relation occurs more likely than a found relation that does not hold at least locally.**
- The considered entry can be also a measured or calculated parameter of the dynamic construction.
- The entry can be a parameter that was computed in the Sketch Editor (e.g., a product of two distances). **If the parameter is 1-dimensional, its name should begin with a lowercase letter. If the parameter is 2-dimensional quantity, its name should begin with an uppercase letter.**
- The considered entry can contain a geometric quantity, e.g.  $d(AB)$ , or **a ratio of two quantities** of the same dimension, e.g.  $d(AB):d(CD)$  or  $Area(ABP):Area(CAP)$ .

- Besides  $d(PQ)$  and  $\text{Area}(PQR)$  you can use a whole range of geometric quantities. The list of all admissible geometric quantities is in Section 4.4.2.
- The options of the command are the same as described in Section 4.2.2. The treatment of the results is the same as described in Section 4.2.3.

The triangle variant of **Observing formulae** is time consuming. Increasing the **strength** and allowing **transformation of variables** significantly increases the computational time. You may consider in such cases to checkmark the **Stop at first option**.

#### 4.4.2 Geometric quantities of the reference triangle

Two symbols are used for distance:

$d(AB)$	distance between the points A and B
$o(ABC)$	oriented distance from the point A to the oriented line BC

The results of the **triangle** variant of **Observe formulae** expressed in terms of certain parameters of the reference triangle. Below is the list of the used parameters. Note that the list is essentially the same as the list in Section 4.3.2. The list is self-explanatory. For example, if PQR is the reference triangle, then  $cA$  means  $r \cdot \cos(P)$ , where  $r$  is the circumradius of the triangle PQR.

a, b, c	length of the sides of the reference triangle
s	half-perimeter of the reference triangle
r	radius of the circumcircle of the reference triangle
ri	radius of the incircle of the reference triangle
Area	area of the reference triangle
S	twice the area of the reference triangle
SA, SB, SC, SW	Conway parameters for the reference triangle
xt(P), yt(P), zt(P)	trilinears of point P
Xb(P), Yb(P), Zb(P)	barycentrics of point P
a, b, c	$2r \cdot \sin(A)$ , $2r \cdot \sin(B)$ , $2r \cdot \sin(C)$ (no need for sA, sB, sC)
cA, cB, cC	$r \cdot \cos(A)$ , $r \cdot \cos(B)$ , $r \cdot \cos(C)$
tA, tB, tC	$r \cdot \tan(A)$ , $r \cdot \tan(B)$ , $r \cdot \tan(C)$
cA2, cB2, cC2	$r \cdot \cos(A/2)$ , $r \cdot \cos(B/2)$ , $r \cdot \cos(C/2)$
sA2, sB2, sC2	$r \cdot \sin(A/2)$ , $r \cdot \sin(B/2)$ , $r \cdot \sin(C/2)$
tA2, tB2, tC2	$r \cdot \tan(A/2)$ , $r \cdot \tan(B/2)$ , $r \cdot \tan(C/2)$
cA3, cB3, cC3	$r \cdot \cos(A/3)$ , $r \cdot \cos(B/3)$ , $r \cdot \cos(C/3)$
sA3, sB3, sC3	$r \cdot \sin(A/3)$ , $r \cdot \sin(B/3)$ , $r \cdot \sin(C/3)$
tA3, tB3, tC3	$r \cdot \tan(A/3)$ , $r \cdot \tan(B/3)$ , $r \cdot \tan(C/3)$
cW, sW, tW	$r \cdot \cos(W)$ , $r \cdot \sin(W)$ , $r \cdot \tan(W)$

r .... circumradius of the reference triangle  
A,B,C .... angles of the reference triangle  
W .... Brocard angle of the reference triangle

### 4.4.3 More examples

#### Example 1

Given is a triangle ABC. Let  $A'$ ,  $B'$ ,  $C'$  be the points of contact of the incircle of ABC with the sides of ABC (Figure 48). The lines  $AA'$ ,  $BB'$ ,  $CC'$  concur in a point P (the Gergonne point).<sup>37</sup>

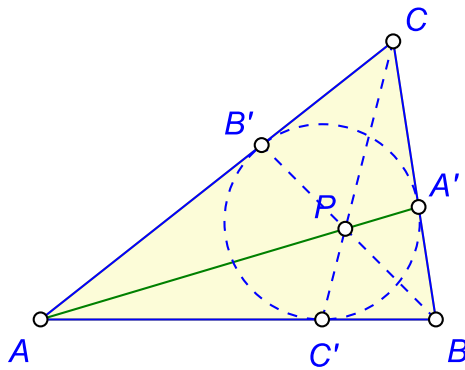


Figure 48

The **triangle** variant of **Observe formulae** suggest several expressions for  $|AP|:|PA'|$  as well as  $|AP|$  or  $|AA'|$ . (The more the strength the more relations are detected.) Here are just some of them:

$$\frac{|AP|}{|PA'|} = \frac{\tan \frac{B}{2} + \tan \frac{C}{2}}{\tan \frac{A}{2}} = \frac{2a(-a+b+c)}{(a-b+c)(a+b-c)}.$$

#### Example 2

Given is an **acute** triangle ABC. Consider the circles  $k_A$ ,  $k_B$ ,  $k_C$  that have as diameters the sides BC, CA, AB of the triangle ABC. What is the radius  $rr$  of the circle  $k$  that contains these circles and is tangent to all of them (Figure 49, left).<sup>38</sup>

<sup>37</sup> OKExamples\OKG\_Plus\Formula\_12.p

<sup>38</sup> OKExamples\OKG\_Plus\Formula\_13.p

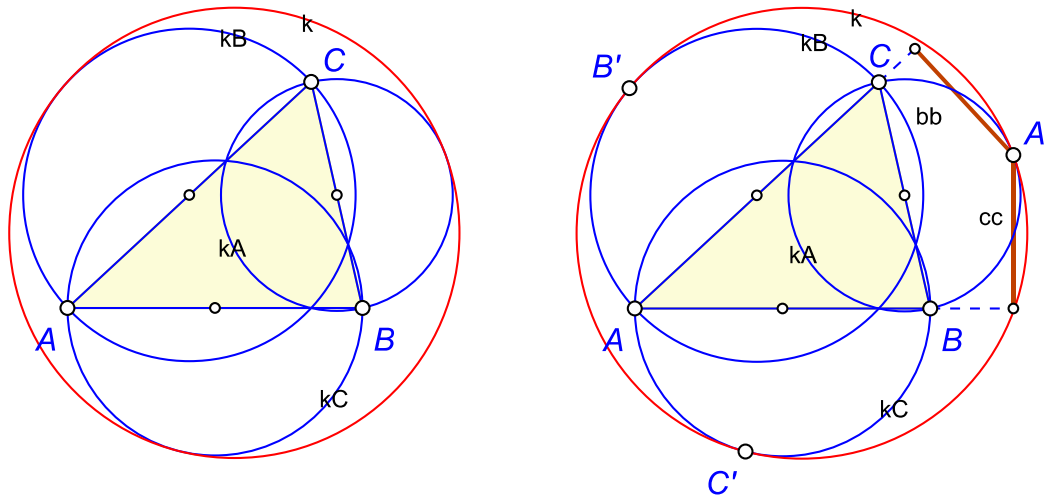


Figure 49

**Observe formulae** for triangles at strength 4 suggests that

$$rr = \frac{S \cdot (2r + ri) + 2 \cdot ri \cdot (SW + ri \cdot (4r + ri))}{2 \cdot (S + ri \cdot (4r + ri))}$$

### Example 3

Continuing the previous construction, let  $A'$ ,  $B'$ ,  $C'$  be the points of contact of the circles  $k_A$ ,  $k_B$ ,  $k_C$  and the circle  $k$ . Furthermore, let  $bb$  and  $cc$  be the distance from  $A'$  to the orthogonal projections of  $A'$  onto  $AC$  and onto  $AB$  (Figure 49, right).<sup>39</sup>

**Observe formulae** suggests that

$$\frac{bb}{cc} = \frac{1 + \sin C}{1 + \sin B}$$

Under this assumption we deduce that  $AA'$ ,  $BB'$ , and  $CC'$  meet at a common point with trilinear coordinates

$$\frac{1}{1 + \sin A} : \frac{1}{1 + \sin B} : \frac{1}{1 + \sin C}$$

The point of concurrence of  $AA'$ ,  $BB'$ ,  $CC'$  is the Paasche point (X1123) of triangle  $ABC$ .

<sup>39</sup> OKExamples\OKG\_Plus\Formula\_14.p



## 5 Generic constructions

### 5.1 Introduction

Consider the following construction (Figure 50): let  $P$  be the orthocentre of a given triangle  $ABC$ . Consider the following three circles: the circle passing through  $B, C, P$ , the circle passing through  $C, A, P$ , and the third circle through  $A, B, P$ . Label the centres of these circles as  $A'$ ,  $B'$ , and  $C'$  respectively<sup>40</sup>. This well known configuration is blessed with several nice properties that can be easily observed with OK Geometry. For example, the lines  $AA'$ ,  $BB'$ ,  $CC'$  concur in a point, a fact that we check with the condition named *Concu\_xxx* (concurrence of three lines). Moreover, the three circles are mutually congruent and the triangle  $A'B'C'$  is homothetic and congruent to the triangle  $ABC$ .

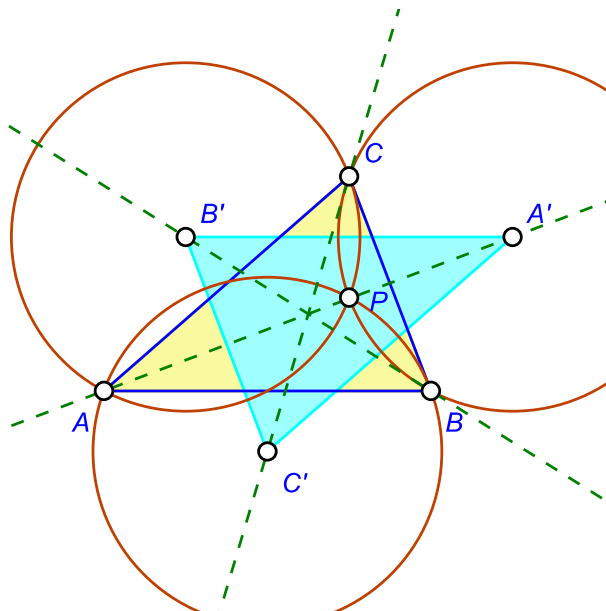


Figure 50

It comes natural to examine constructions similar to the one presented. The point  $P$  could take the place of the centroid of  $ABC$  or circumcentre of  $ABC$  or some other centre of the triangle  $ABC$ . Also, instead of circumcircles of triangles  $BCP$ ,  $CAP$ ,  $ABP$ , the three circles could be incircles, 9-point circles or some other circles related to the aforementioned triangles. Perhaps such modified constructions have some nice properties, but it would be an annoyance to construct all of them one by one. The generic constructions were designed to make life easier in such cases.

Note that the initial configuration was obtained as a sequence of operations, starting from the three vertices  $A, B, C$ , the lines connecting each pair of them, and so on. Two operations are particularly important:

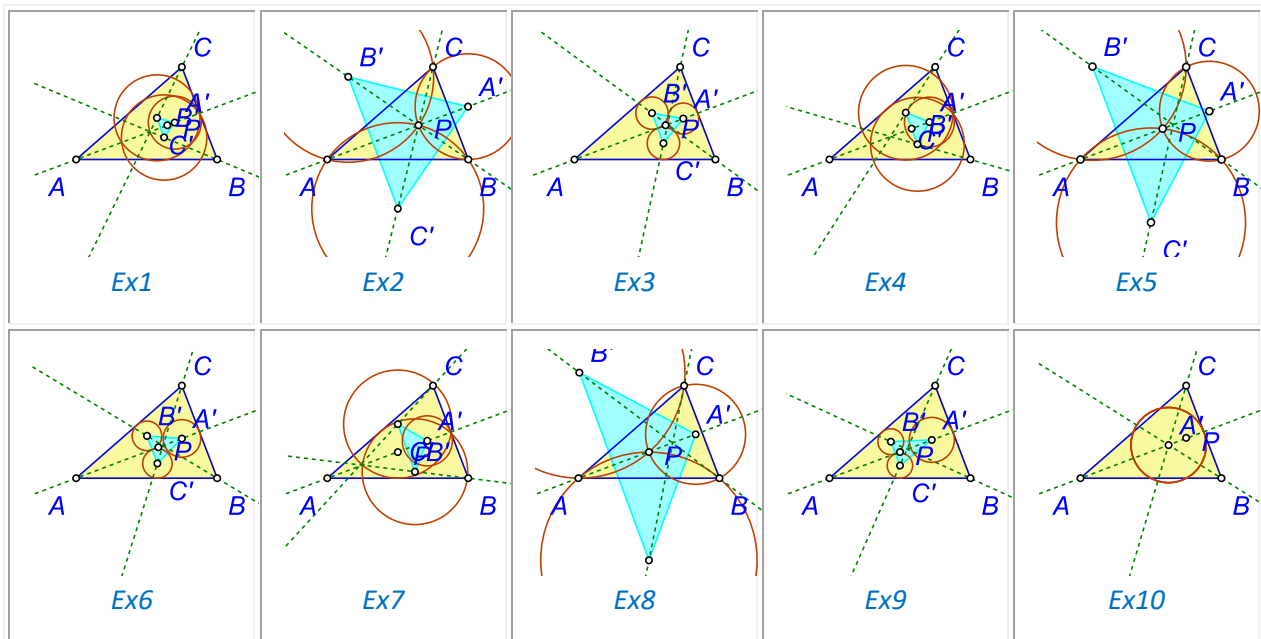
1. Instead of the (complex) operation 'set a point as the orthocentre of a given triangle' we want something like 'set the point  $P$  as each of the following points of a given triangle  $ABC$ :

<sup>40</sup> OKExamples\OKG\_Plus\Generic\_01.p

incentre (ETC centre X1), centroid (ETC centre X2), circumcentre (ETC centre X3), orthocentre (ETC centre X4), 9-point centre (ETC centre X5)'. What we want is a **generic operation** or **rule**. You can think of a rule as a 'union' of functionally isomorphic operations, that is, a union of operations that construct the same type of object with points as arguments. Only one operation of a rule is active at a time, and it is easy to switch between the operations of a rule. In our construction (and this is easy to do in OK Geometry) we would replace the operation 'P is orthocentre of ABC' with a rule, say Pobject1, to obtain 'P results from the rule Pobject1 on the points A,B,C'.

2. Instead of the (complex) operation 'construct a circle as the circumcircle of a given triangle' we want something like 'set a circle as each of the following circles of a given triangle: circumcircle, incircle, 9-point circle'. We need a **rule**, say Cobject1, that contains three operations, each yielding some circle from three given points. We would apply the rule first to points B,C,P, then to points C,A,P, and finally to points A,B,P. The rule Cobject1 would perform the same operation (circumcircle, incircle, or 9-point circle) in all three cases.

As we will see, it is easy either to construct and use new rules or replace an operation in a construction with a rule – in both cases we get a **generic construction**. A generic construction consists of an initial construction called the **base example** and one or more **rules** or **generic operations** attached to some operations in the base example. When a rule is applied, all operations in the base example to which the rule is attached, are modified according to the rule. The constructions obtained by applying rules are called **examples**. Examples are de facto ordinary constructions. In our initial construction, we have attached to the operation orthocentre (of ABC) the rule Pobject1, which consists of 5 variants (operations). Also, we have attached to the operations circumcircle (of BCP, CAP, ABP) the rule Cobject1, which consists of 3 variants. If we activate both rules, we obtain  $3 \times 5 = 15$  examples of our generic construction. The examples are shown in Figure 51. Note that the example Ex\_10 is missing because it is degenerated. In fact, all three circles coincide.



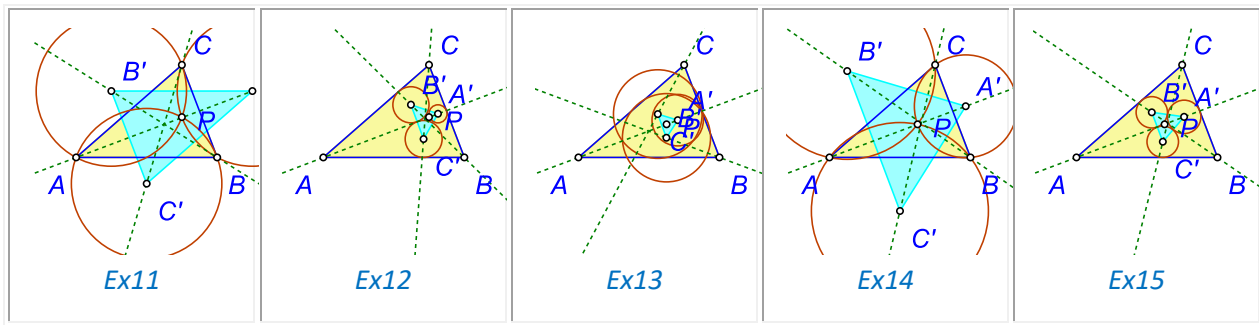


Figure 51

Generic constructions are easily obtained, either by modification of an ordinary construction or by direct construction. A generic construction can contain dozens, hundreds or even thousands of examples.

Generic constructions allow you to:

- ✓ **simultaneously generate a large number of examples** (constructions);
- ✓ **easily access to any example of a generic construction**, examples can be observed, exported, etc., just like ordinary constructions;
- ✓ **automatically search examples of a generic construction** in order to find examples with specified properties;
- ✓ **automatically create projects from examples** of a generic construction;
- ✓ **automatically perform triangle analysis on all or some examples** of a generic construction;
- ✓ **automatically observe formulae on all or some examples** of a generic construction.

In simple terms, a generic construction is an ordinary construction, in which one or more **rules** (generic operations, 'unions of operations') are attached to selected operations (construction steps). A **generic construction** contains several constructions, called **examples**, obtained by applying various operations in rules. When no rule is active, the generic construction appears visually as the **base example** and behaves like an ordinary construction. The concept of rule (generic operation) is explained in detail in Section 5.3. In the next section we give an overview of the commands related to generic constructions and the way they are used.

## 5.2 Working with generic constructions (an overview)

All commands related to generic constructions are shown in Figure 52 and Figure 53.

Generic constructions as files are stored like ordinary constructions (files with extension .p) or projects (files with extension .pro). The generic construction of the introductory example is in the file *Generic\_02.p*<sup>41</sup>. When you open a generic construction, you see the base construction appearing like an ordinary construction. The only indication that it is a generic construction is the button **Gen** which turns reddish (otherwise it is gray and inactive), Figure 52.

Generic constructions can be accessed at any time (except when icons of a project are displayed).

<sup>41</sup> OKExamples\OKG\_Plus\Generic\_02.p

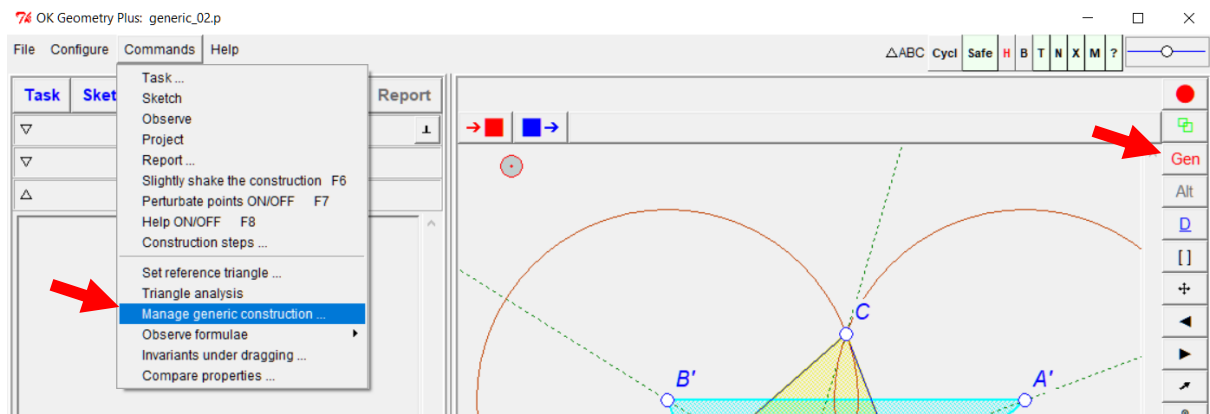


Figure 52

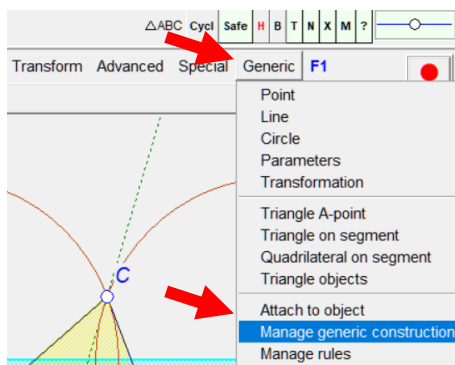


Figure 53

All commands for creating generic constructions are located in the group *Generic* of the Sketch Editor. There are three groups of commands: 1. generic operations for creating new objects (Point, Line,...), 2. generic operations for non-standard objects and triangle objects, 3. commands for handling generic constructions.

### 5.2.1 Attaching rules to operations

We first show how to attach a rule (generic operation) to an operation of an ordinary construction (or the base example of a generic construction). We start from an ordinary construction<sup>42</sup> described in the introduction (Figure 50). You can create the initial construction in OK Geometry, import it from a system of dynamic geometry, or use the supplied file *Generic\_01.p*.

First, we attach to the point P, which is now the orthocentre of ABC, a rule consisting of triangle centres X1 to X5. Here are the steps:

1. Enter in the Sketch Editor mode.
2. Click the command *Generic/Attach to object*. Note the information line just above the displayed construction – OK Geometry expects a point, line or circle.
3. Click on the point P. A point rule form appears (Figure 54). Since we are creating a new rule we choose the name *Pobject1 (new)* and immediately declare the operations in the rule, i.e., ETC triangle centres 1-5. Note the “3:” in front of the operation names for ETC centres – 3 means that the operation takes 3 points as arguments. We also checkmark the points A, B, C

<sup>42</sup> OKExamples\OKG\_Plus\Generic\_01.p

used by the operations. The order in which the points are marked generally matters, but not in our case. Note that  $A'$ ,  $B'$ ,  $C'$  are not listed among the points, because in the construction they depend on the point  $P$ . Then click the OK button. (Do not bother of other functions in the form.)

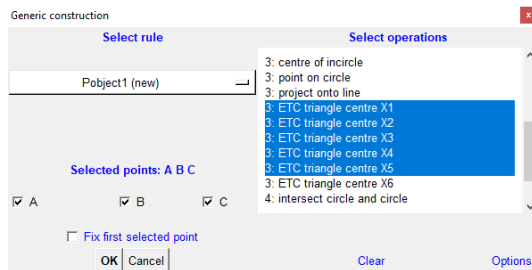


Figure 54

Now consider the circle through  $B, C, P$ , that is, the circumcircle of the triangle  $BCP$ . As described above, use the command *Generic/Attach to object*, then click on the circle through  $B, C, P$ . In the form that appears select the rule *Cobject1 (new)* and then select the operations: 9-point circle, circumcircle, and incircle of a triangle. Be sure to checkmark the points  $B, C, P$  (the order of checking is not important in this case). Then click OK.

The circles through  $C, A, P$  and through  $A, B, P$  must obey the same rule as the circle through  $B, C, P$ , that is, all three circles must be incircles of the respective triangles, or circumcircles, or 9-point circles. When we attach a rule to the circle through  $C, A, P$  we do not declare a new rule, but choose the previously declared rule *Cobject1* (Figure 55). We proceed similarly for the circle through  $A, B, P$ .

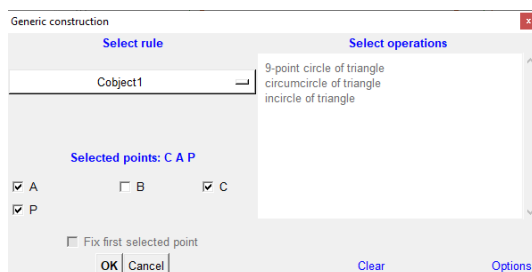


Figure 55

Our generic construction is ready. You can save it to a file as you save ordinary constructions. This generic construction is also available in the file *Generic\_02.p*.

## 5.2.2 Accessing examples of generic constructions

The **Gen** button indicates whether the current construction is generic. A generic construction of the introductory example is available for download as file *Generic\_02.p*<sup>43</sup>.

There are three ways to activate the examples in a generic construction (see Figure 52 and Figure 53):

- Press the **Gen** button on the toolbar to the right. This is the usual way, it is always accessible.

<sup>43</sup> OKExamples\OKG\_Plus\Generic\_02.p

- Use the command *Commands/Manage generic constructions*.
- In Sketch Editor click the command *Generic/Manage generic constructions*.

A form for managing examples appears (Figure 56). For convenience we refer to this form as **Examples- form**.

**Important notice.** A generic construction cannot be modified when the Examples-form is active. When you exit the Examples-form the generic construction is as it was when you accessed the form.

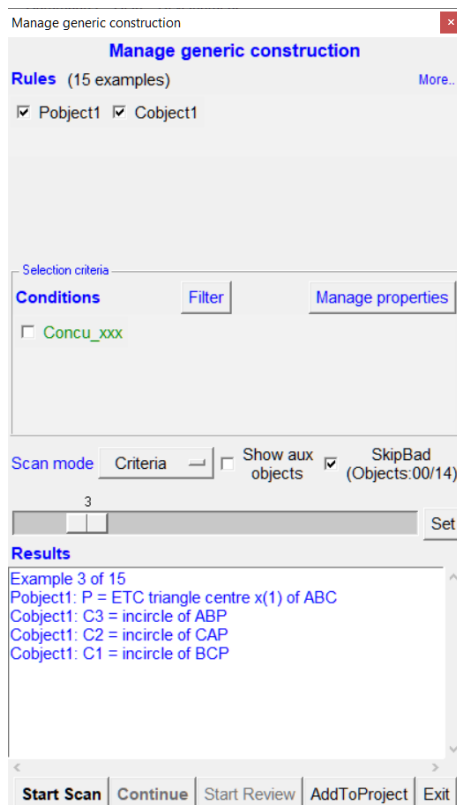


Figure 56

To access the examples, do the following:

1. Checkmark the rules you want to activate. In our case we have marked both rules, i.e. *Pobject1* and *Cobject1*. We can see that the rules give rise to 15 examples.  
**Note.** The order in which the rules are marked determines the sequence order of examples. In our case we have marked *Pobject1* first and then *Cobject1*.  
**Note.** If a rule is not activated, the corresponding operation from the base construction is used.
2. Leave the middle part of the form (Conditions, etc.) blank. These entries are filled only for scanning through the examples in search of examples with particular properties.
3. Select **Example** as the Scan mode.  
 You can choose whether or not to display auxiliary objects that are automatically created in examples. In our case, a displayed centre of a triangle colours the corresponding triangle green. Check or uncheck according to your preference.  
 We shall explain the **SkipBad** option later.

4. Press the **Start Scan** and then repeatedly the **Continue** button. An example is displayed at each step.
5. You can also position the slider at a number and then press the **Set** button to display the corresponding example.
6. **The example you see at each step is an ordinary construction. You can do with the example what you can do with any construction: zoom it, drag points, delete objects or add new objects, observe the properties, save the construction as an ordinary construction etc. However, unless you save the example to a separate file or icon, all changes to the example will be lost when you switch to another example or close the Examples-form.**
7. The examples in generic construction can be organised into a project by pressing the **AddToProject** button. The examples are stored as icons (ordinary constructions) with names with suffix `_x`, while the generic construction is stored under the name with suffix `_all`.

### 5.2.3 Creating a selection of examples

In our introductory example we have (observationally) checked whether the lines  $AA'$ ,  $BB'$ ,  $CC'$  concur in a point. This was done using the command *Advanced/Check property/ConcurrentL* in the Sketch Editor. The resulting parameter *Concu\_xxx* turned to be (True). We wonder in which of the 15 examples of the generic construction (file *Generic\_02.p*) the condition *Concu\_xxx* holds as well. Here is how to proceed.

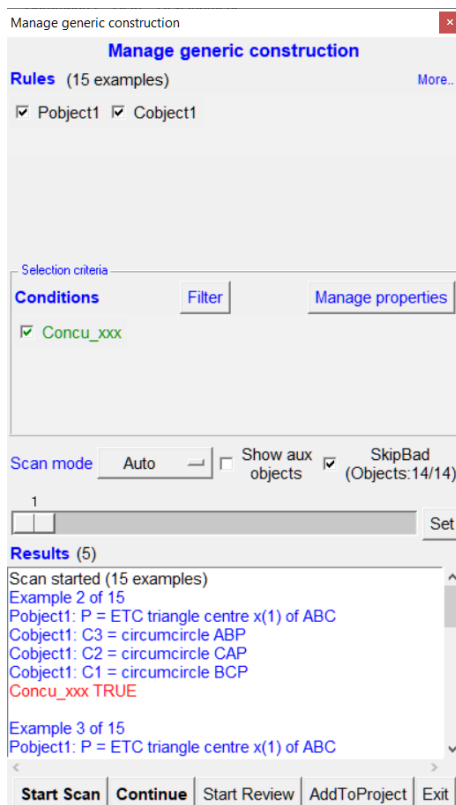


Figure 57

1. In the Examples-form checkmark the rules you want to consider (Figure 57, see previous section).
2. Set the selection criteria for examples. Under the **Conditions** you see the conditions (checks) that were made in the base example of the generic construction. Checkmark the conditions you consider as criteria for the selection of examples. At this point you could consider also properties as criteria, but for now this is not the case. When scanning, the system looks for examples in which **at least one of the selected criteria** is fulfilled.
3. For the Scan mode choose **Criteria** or **Auto**.  
You can choose whether or not to show auxiliary objects that are automatically created in the examples.  
If **SkipBad** is marked, scanning will not consider examples that contain fewer objects than the base example. We recommend to always checkmark **SkipBad** entry when scanning, as a smaller number of objects means that some objects could not be constructed or that some points coincide.
4. Click on the **Start Scan** button. If you selected the **Auto** mode of scanning, wait until the scan is finished and the result is displayed in the results area. In our case 5, examples were found. If you selected **Criteria** as scanning mode, the examples that meet at least one of the marked conditions will be displayed one by one as you click on the **Continue** button.
5. To review the examples in the obtained selection, click on the **Start Review** button and then repeatedly press the **Continue** button.
6. **The example you see at each step is an ordinary construction. You can do with the example what you can do with any construction: zoom it, drag points, delete objects or add new objects, observe the properties, save it as an ordinary construction, set it as an icon, etc. However, unless you save the example to a separate file or icon, all changes you make to the example will be lost as you switch to another example or close the Examples-form.**
7. The examples in the obtained selection can be organised into a project by pressing the **AddToProject** button. The examples are stored as icons (ordinary constructions) with names with suffix **\_x**, while the generic construction is stored under the name with suffix **\_all**.

The Sketch Editor command *Advanced/Check property* provides a variety of geometric properties. The obtained condition can be composed with the command *Number/Compose conditions*. You can thus check the logical conjunction, disjunction, negation of conditions (use the operators and, or, not). Furthermore, the *Advanced/Check property/Other* command provides additional properties that you may find useful (e.g. the True property, which is always true).



### 5.2.4 Fast scan of conditions criteria

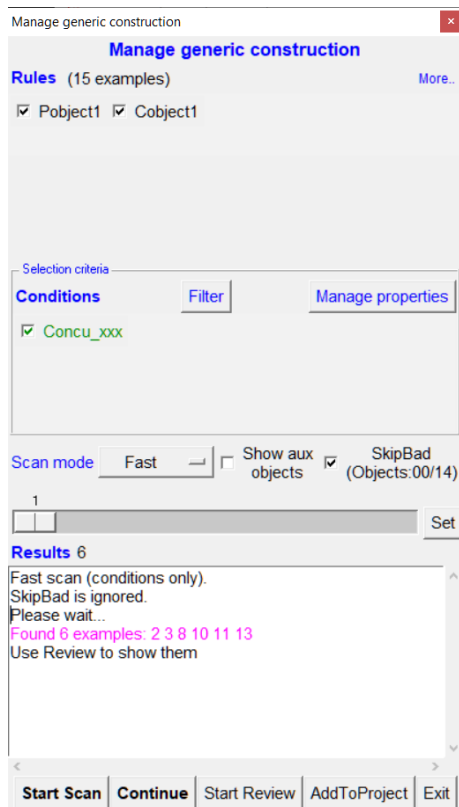


Figure 58

Scanning examples can take a lot of time. If the generic construction contains hundreds or thousands of examples, it is better to choose **Fast** as Scan mode. The method considers only conditions as criteria (properties are ignored).

You proceed as described in the previous section. However, after you press the **Start Scan** button (and wait a bit) you can read in the Results section which examples satisfy at least one of the specified conditions. You can review these examples by pressing the **Start Review** button and then repeatedly the **Continue** button.

### 5.2.5 Observing properties in examples

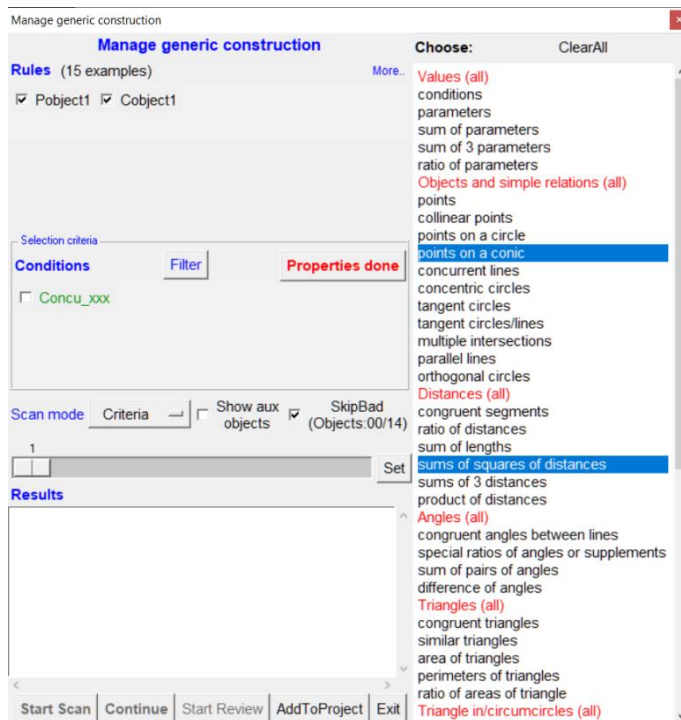


Figure 59

It is possible to scan all the examples of a generic construction to find which of them a certain property has been observed. The procedure is, basically, the same as for observing whether certain conditions are met, except that instead of the conditions you specify the properties to be observed.

To specify the properties to observe, click on the **Manage properties** button. In the list of properties, select the properties you want to observe. In our case we selected 2 properties: if any 6-tuple of points in an example lay on a conic, if any two or more pairs of squares of the distances between points in the example add up to same sum. As you can see, the properties are divided into groups. When you select a property coloured red you select all the properties in the group below it.

Once the properties are selected, click on the **Properties done** button (otherwise you cannot proceed).

For the scanning method you can select either **Criteria** (i.e. stop when a criterion is observed) or **Auto** (i.e., scan to the end). Whenever an example is found or reviewed, you can see the properties found in the **Observed properties** section. Figure 60 shows the display as OK Geometry observed in Example 11 both observed properties.

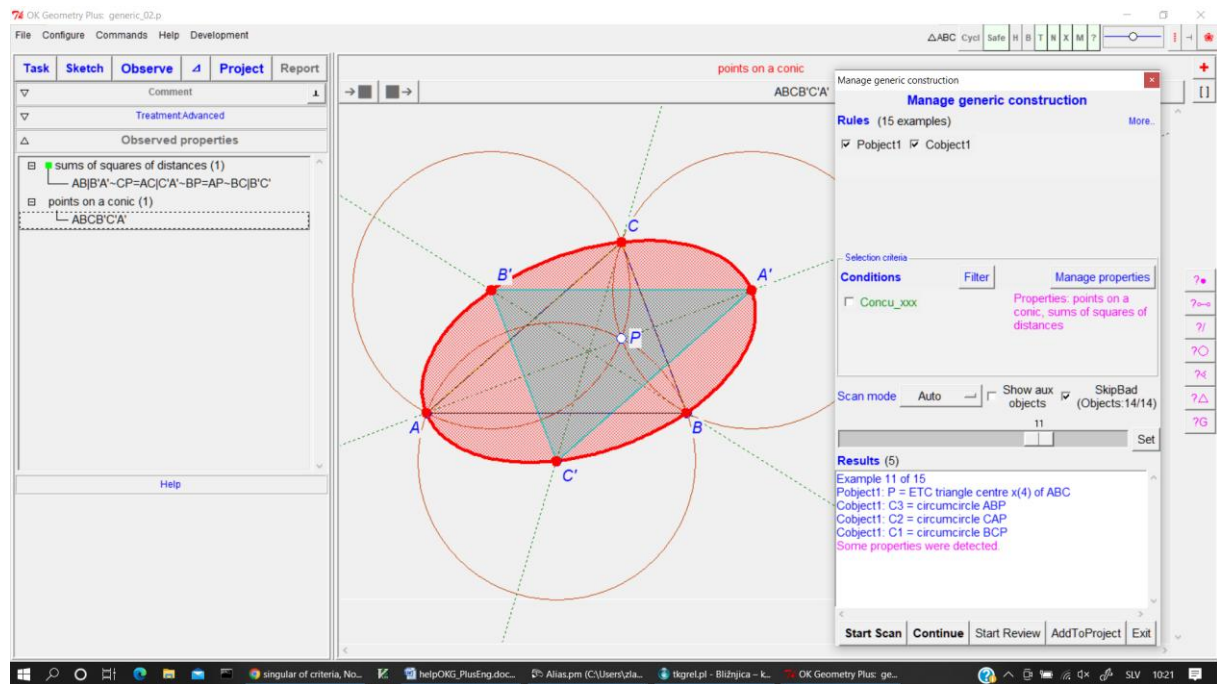


Figure 60

The observation of properties in examples is a powerful tool, provided one takes into account some important facts.

1. There is a big difference between observing properties and observing conditions. A condition always refers to a specific property of specific objects whereas properties consider all objects. To illustrate the difference, consider the case of concurrence of lines in our generic construction. All examples contain 7 labelled points each that determine, in general, 21 lines, which give rise to 1330 triplets of lines. Selecting the 'concurrent lines' property means that you observe the concurrency of all triplets in all examples of the generic property. In contrast, selecting the condition that the lines  $AA'$ ,  $BB'$ ,  $CC'$  are concurrent requires only one observation per example. So **observing properties is slower than observing conditions and much slower than fast observing of conditions**.
2. The speed of observation of properties in examples depends on: the number of properties observed, the type of properties observed, and the number of (labelled) points involved. Observing a lot of properties at the same time is unwise, since each example will almost certainly have one or another property. Therefore, it is reasonable to **limit yourself to essential properties**. The best speed improvement is obtained by **restricting the observation to the essential (labelled) points**. This is done by specifying the points to consider in the section **Treatment|Consider only points** in the observation pane.
3. The properties in the list of observations are to some extent self-explanatory. Nevertheless, there are some peculiarities that are not self-evident:
  - a. The properties based on geometric quantities or numerical values (e.g., parameters, congruent segments, congruent angles, area of triangle, congruent circumcircles, congruent incircle) take into account not only equality of the quantities but also many relationships between triplets of quantities (e.g. radius of a circle is equal to the sum of radii of two other circles)

- b. Properties based on the ratio of two geometric properties (e.g., ratio of parameters, ratio of distances, ratio of angle sizes, ratio of areas of triangles, ratio of radii of circumcircles, ratio of radii of incircles) consider not only the equality of two ratios, but also whether a ratio is observed to be constant, as well as some relationships between triplets of ratios.
  - c. The property 'points' observes whether a point is a 'simple' centre of any triplet of labelled points in the configuration.
- 4. If it turns out that too many properties of examples are found and listed during the scan, we can also reduce their number (besides reducing the set of observed labelled points) also by applying a filter to the listed instances of properties. This is the purpose of the **Filter** button.

To summarise this section, let us again consider the generic construction in file *Generic\_02.p*. Let us examine the position of the point P with respect to the triangle A'B'C'. We describe three ways of how to proceed.

1. The simplest way is to display all 15 examples one by one (as described in Section 5.2.2). For each example we analyse the position of the point P in the reference triangle A'B'C' (the command **Special/Analyse object wrt. triangle**). We can do this way because the number of examples is not excessive.
2. We can do an automatic scan of all the examples. Since the property 'points' refers to simple centres in triangles with labelled points as vertices, we observe the property **points** with the scanning option **Auto** or **Criteria**. Furthermore, in the **Treatment** section of the observation pane we choose to observe only points A'B'C'P.
3. A more sophisticated method of triangle analysis in examples is explained in Section 5.5.

### 5.3 Rules (generic operations)

In Section 5.1 we introduced the concept of **rule (generic operation)** as something analogous to a geometric operation in a construction, as a 'union' of operations. An example of geometric operation is the midpoint (of two points). We can apply this operation several times in a construction to different pairs of points. Similarly, in a generic construction a rule can be applied several times to sets of points. Only one operation of the rule can take place at time – and that same operation acts everywhere the rule is applied

Figure 61 illustrates a rule that contains three operations:

- midpoint (of two points),
- the third vertex of a positively oriented equilateral triangle (based on two points)
- a free point on a line (through the two points).

Each operation of the rule acts on a pair of points and creates a new point. The rule was applied to the points A,B and then to B,C.

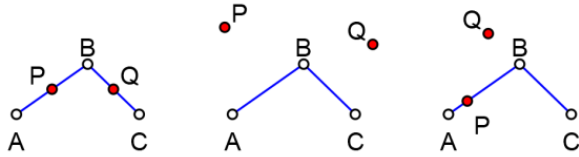


Figure 61

The rule gives rise to three examples, one for each operation in the rule. The points created with rules have a peculiarity. If they are free points, they change position but they cannot be dragged with the mouse. Thus, the red points in the third example in Figure 61 are free points on the lines AB and BC. They change position each time the example is displayed, but they cannot be dragged.

The analogy between the rules and operations is only partial. In fact, a rule consists of three components:

- The first component is the set operations that can take place when a rule is applied. The operations involved in the rule act **only on a certain number of points**. An operation, like ‘construct a line through a given point and parallel to a given line’ cannot be included in a rule. On the other hand, the operation ‘construct a line through a given point and parallel to a line through two given points’ is allowed.
- The second component is the number of points on which a rule acts. This number is specified when a rule is declared. The same rule can be applied several times in a generic construction on different sequences of points, each consisting of the same number of points.
- The third component is the role of the order of points in a sequence on which a rule operates and in particular the role of the first point (the fix-first-point attribute).

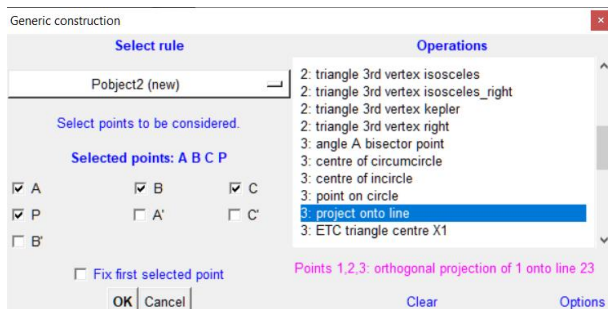


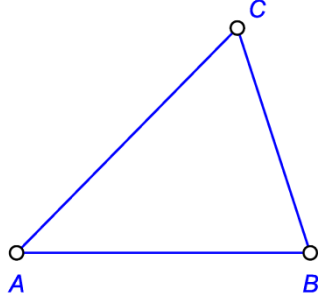
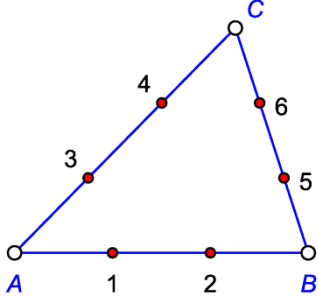
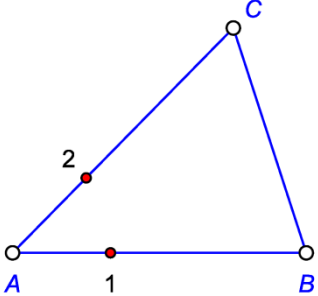
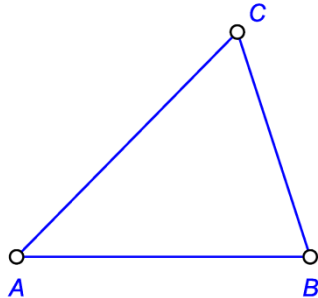
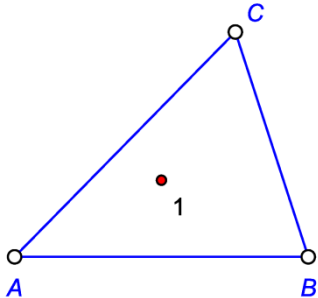
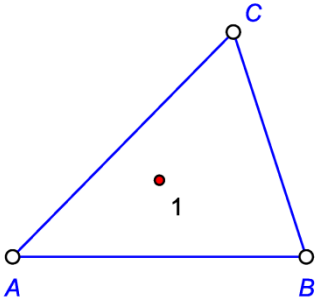
Figure 62

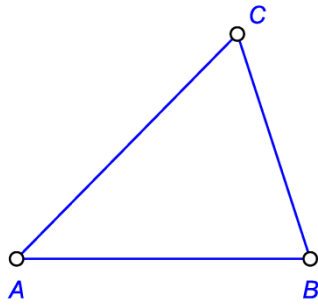
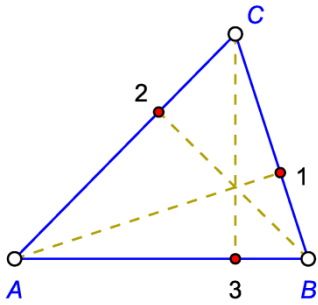
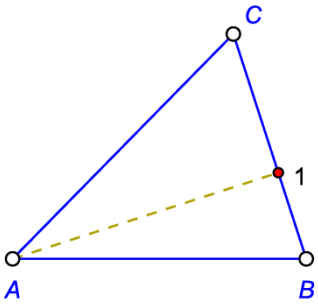
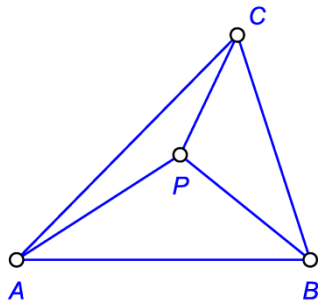
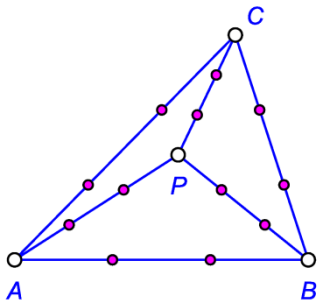
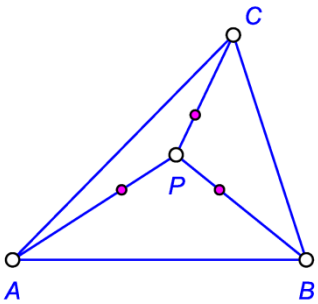
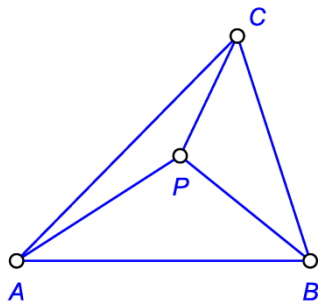
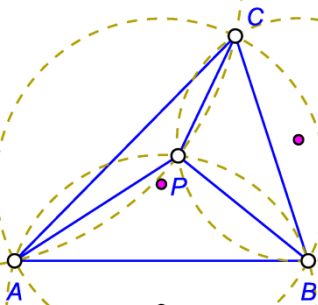
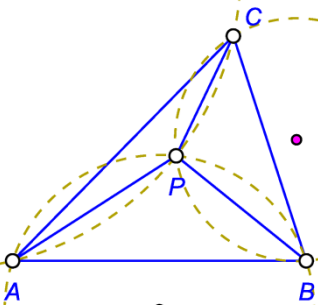
The declaration of a new rule occurs simultaneously with its first application. To declare a rule, we must specify the operations to be included in the rule, the points subject to the rule, and the fix-first-point status. Each subsequent occurrence of the rule will contain the same operations, the rule must be applied to the same number of points, and the fix-first-point status cannot be changed either. Figure 62 shows the process of declaring a new rule Pobject2. The ‘project onto line’ operation has been selected as one of the operations. Note the ‘3:’ in front of the operation name – it indicates that the operation uses three points (the number of the selected points of the rule may be greater). Also note the magenta explanation of the operation – if the operation is applied to three points, then the 1<sup>st</sup> point is projected orthogonally onto the line passing through the 2<sup>nd</sup> and the 3<sup>rd</sup> point.

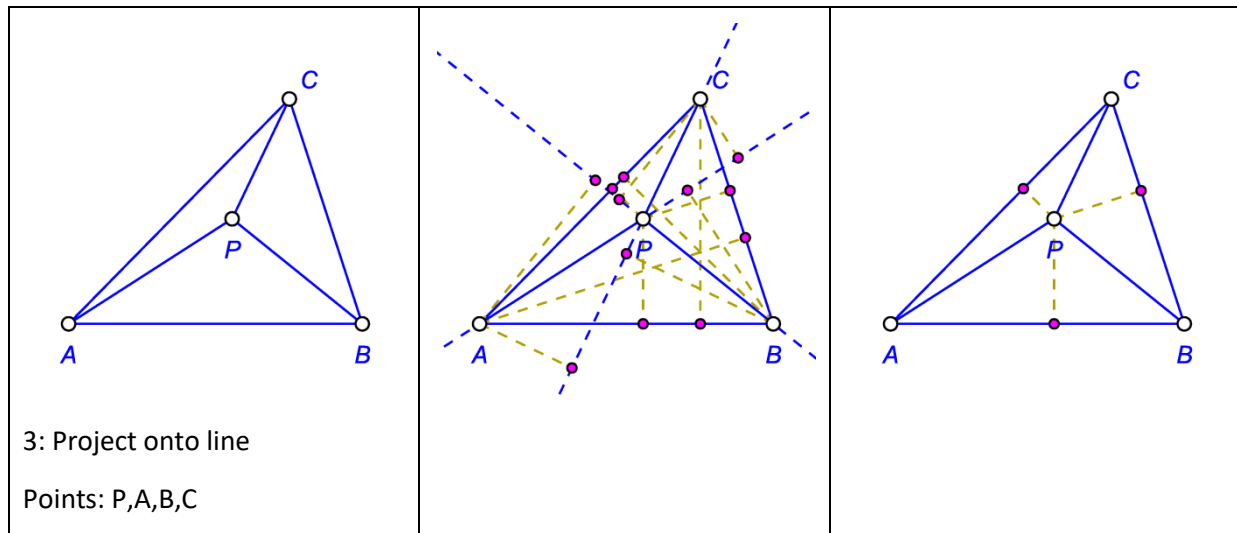
Although one needs not to worry about the components in simple cases, we explain the role of the components with some examples.

Assume we have a rule that yields an object. The rule consists of several operations to be applied to a set of  $n$  points. Let us focus on one operation within the rule. Suppose the operation yields the resulting object taking  $m$  points ( $m \leq n$ ). The operation within the rule acts on every  $m$ -tuple of  $n$ -points and all arrangements within the  $m$ -tuples (i.e. on all permutations of  $m$ -points out of  $n$ -points) yielding an object. Arrangements that are redundant, according to the meaning of the operation, are ignored. By setting the fix-first-point attribute a restriction is imposed: The first point in the list of selected points (of a rule) is also the first point of all considered permutations of points (for all operations contained in the rule).

We clarify the role of sequence of points with some examples:

Configuration, operation, and the sequence of points	Yielded objects within rule if fix-first-point is OFF	Yielded objects within rule if fix-first-point is ON
 <p>2: Point at 1/3 of segment Points: A,B,C</p>		
 <p>3: Centre of circumcircle Points: A,B,C</p>		

 <p>3: Project onto line Points: A,B,C</p>		
 <p>2: Point at 1/3 of segment Points: P,A,B,C</p>		
 <p>3: Centre of circumcircle Points: P,A,B,C</p>		



An operation within a rule generally results in a sequence of objects (only one created object appears in any example). To make rules manageable, it is better to use within a rule only operations that act on the same number of points. Also, a rule should require the same or a slightly larger number of points than the operations contained in the rule.

The rules in the Generic menu are divided into two or three groups. In the next subsections we give an overview of the rules. All rules create new objects from points. The names of the rules are fixed, the first letter of the name indicates the type of objects created by the rule (P – point, L – line, C – circle, T – triangle, Q – quadrilateral, S-conic). For example, the names of the rules for creating points are Pobject1, Pobject2, Pobject3, and so on. An exception are the rules ETC1, ETC2, ETC,.. for ETC triangle centres.

The commands in the first two parts of the **Generic** menu are used just like other commands for creating constructions. The commands appear like commands for creating a point, line, or other object. In fact, they create objects in the current construction, which is the base example, but at the same time they declare and/or apply a rule in the generic construction. For example Figure 61 illustrates a simple acting of a rule containing three operations. When the rule is declared or applied to a set of points, one of the operations contained in the rule also becomes a corresponding operation in the base example, which is also displayed in the construction.

### 5.3.1 Rules for creating simple objects

The commands for the rules for creating points (*Gpoint*), lines (*Gline*), and circles (*Gcircle*) are located in the top part of the **Generic** menu commands. When one of these commands is activated, a form appears (Figure 63). We will take a closer look at the form for creating a rule for points, since the forms for creating lines and circles are pretty self explanatory.



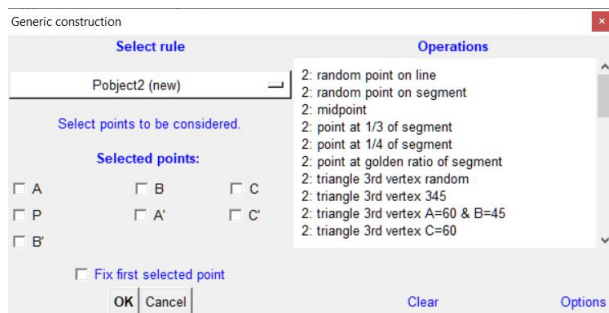


Figure 63

In the list of operations that can be included in the rule for points there are operations (e.g. midpoint) that do not require an explanation and groups of operations that do.

**Point at x of segment.** The values listed are  $1/2$  (midpoint),  $1/3$ ,  $1/4$ , and  $(\sqrt{5}-1)/2$  (golden ratio). You can add more values to the list by clicking *Options/Divide segment values* (lower right corner of the form).

**Triangle 3<sup>rd</sup> vertex ...** . Given two vertices (green coloured points in Figure 64), this group of commands creates the 3<sup>rd</sup> vertex (red coloured point) of a positively oriented triangle. The position of the third vertex is described (up to direct similarity) with one or two conditions in terms of parameters shown in Figure 64. Some examples: 'A=60 & B=60' positions the 3<sup>rd</sup> vertex of an equilateral triangle; 'C=90' positions the 3<sup>rd</sup> vertex on a semicircle whose endpoints are the given points; 'a:b=2 & A=72' or 'a:b:c=5:6:7' are also acceptable descriptions. Some common names are also allowed as descriptions of the resulting triangle: equilateral, 345 (i.e. a triangle similar to a right triangle with sides 3,4,5), isosceles, etc. The list of Triangle 3<sup>rd</sup> vertex operations can be modified by deleting or adding new descriptions. You can modify the list of commands by clicking *Options/Triangle 3rd vertex* (lower right corner in the form) and filling the form that appears.

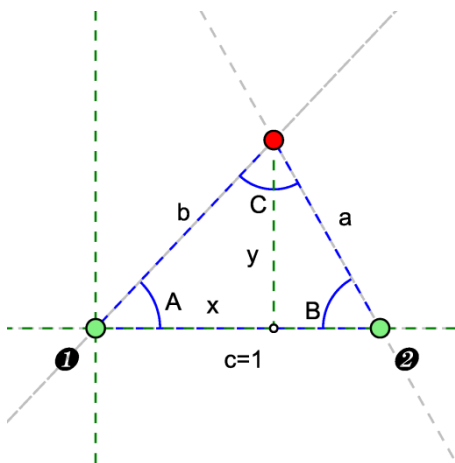


Figure 64

**ETC triangle centres ...** . In the list of operations, there is a group of operations that yield some ETC centres from three given points. You can modify the list using *Options/ETC triangle centres* (lower right corner of the form). In the form that appears specify the numbers of ETC centres to appear in the list. Note that a full list of ETC centres is available in another group of rules (see 5.3.5).

**Quadrilateral centres ...** . The list contains a few QA-quadrilateral centres (i.e. centres obtained from vertices irrespective of their order). The generic examples thus do not permute the vertices of a quadrilateral. Note that a full list of quadrilateral objects is available in another group of rules (see 5.3.6).

The rules for creating lines and circles are self-explanatory.

It is also possible to create rules for numeric parameters. These rules are simply lists of numbers or numeric expressions.

### 5.3.2 Rules for transforming simple objects

The **GTransformation** rule creates new points (lines, circles) from existing originals using a set of selected transformations. The rule leaves the original untouched, merely adding a new object of the same type as the original to the construction. The operations (i.e. transformations) in the rule are always defined in terms of some points.

To declare a transformation rule

1. Activate the *Generic/GTransformation* command.
2. Select the object (point, line, circle) to be transformed.
3. A form appears (Figure 65), the name of the declared rule is Ptransf1, Ptransf2,... for points, Ltransf1, Ltransf2,... for lines, Ctransf1, Ctransf2,...for circles.
4. Fill in the form and press OK. A new object appears in the base construction.

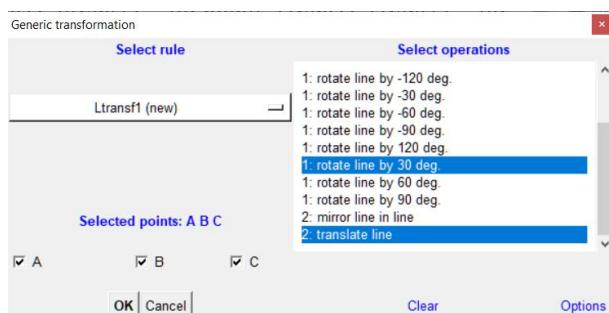


Figure 65

We first comment on the form for declaring rules for transforming lines. Recall that the form appears after a line was selected in the base construction. In our case (Figure 65), the base example clearly contains points A,B,C. You need to select the operations and the points that define the operations. In Figure 65, three points have been selected and two operations (rotation by 30 degrees and translation). Rotation by 30 degrees is defined by 1 point (see '1:' in front of the name of the operation). Thus the generated line in the examples is the original line rotated by 30 degrees around A, around B and around C. In further examples (resulting from the same rule) the generated line is the translation of the original line by vectors AB, BA, AC, CA, BC, CB. Altogether, the rule produces 9 lines, which are transformations of the original line.

In the list of operations for transformation rules, besides self-explanatory entries, there are two groups of transformations: 'rotate by x degree' and 'dilatate by factor x'. The values x in the list can

be modified by setting the desired values with the commands *Options/Rotation angles* and *Options/Dilatation factors* (lower right corner of the form).

The rules for transforming points have a peculiarity. Suppose the base construction contains a point P and points A and B. We declare a rule for transforming of a point and choose the point P. The point to be created will be thus a transformation of P. Consider now the form that appears.

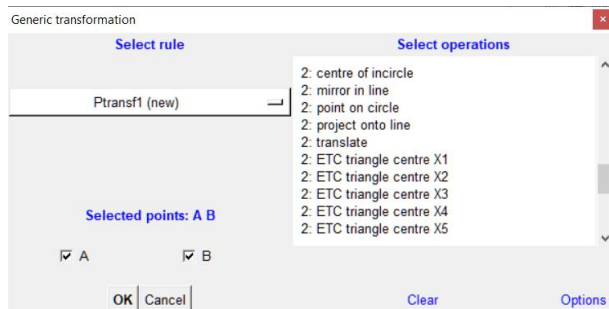


Figure 66

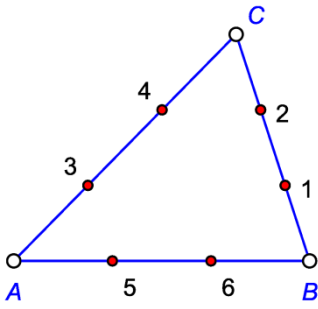
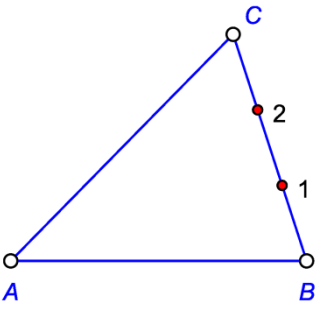
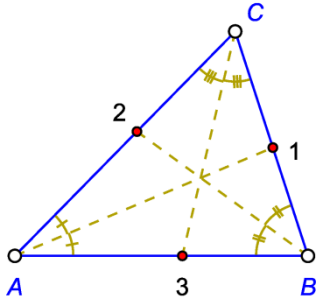
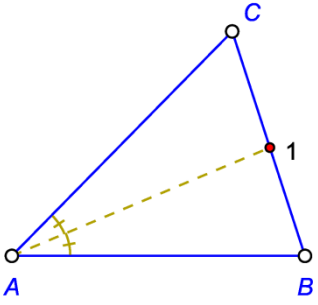
Among the listed operations you can see some ‘standard’ transformations. For example, the entry ‘2: translate’ in our case would yield 2 examples: the point P translated by AB and P translated by BA. On the other hand, observe the entry ‘2: centre of incircle’. Centre of incircle is usually not considered a transformation, it is rather considered an operation on 3 (not 2) points. In such cases the selected point (to be transformed) is considered the first argument of the operation. The transformed point P in this case is thus the incentre of the triangle PAB. Mixing geometric operations and transformation may appear confusing, but on the other hand enables the declaration of richer rules.

The next group of rules are all defined on a fixed number of points.

### 5.3.3 Triangle A-points

The rules named Apoint1, Apoint2,... contain operations that create points in a triangle, often with respect to only one of the vertices of the triangle. These rules can only be applied to triplets of points. As with any rule, the operations in the rule are applied to all permutations of the points in the triplets. So a single operation in the rule will result in up to 6 different points in respective examples. If the first point of the triplet is fixed, a single operation gives rise to 1 or 2 points.

Here are two examples of A-operations:

Configuration, operation, sequence of points	Yielded objects within rule if fix-first-point is OFF	Yielded objects within rule if fix-first-point is ON
Points: A, B, C Operation: Point at 1/3 of A-side		
Points: A, B, C Operation: Angle A bisector point		

The list of operations for triangle A-points contains common operations for points, perhaps modified with respect to a specific vertex. Some operations are rather specific to triangles. In addition to triangle centres (the list can be modified with *Options/ETC triangle centres*) the list of operations contains, relative to a vertex, the pedal point, the Cevian point, the circumcevian point, and the circlecevian point of the included triangle centres (see Figure 67). In this way, a rich variety of operation can be contained in a single rule.

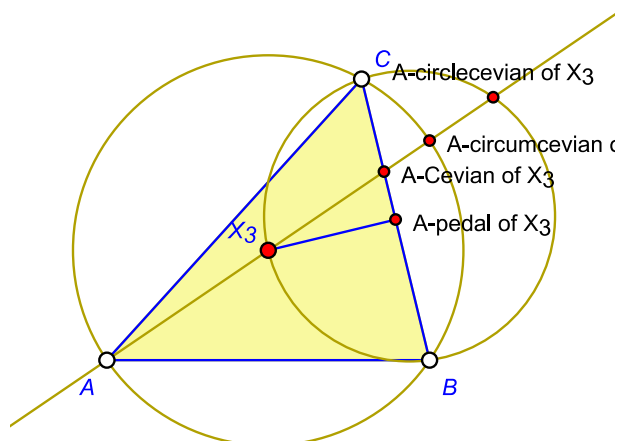


Figure 67

### 5.3.4 Rules for triangles and quadrilaterals on a segment

The *Generic/Triangle on segment* command creates a rule that yields various types of positively oriented triangles with two given vertices. The rules for triangles are named Tobject1, Tobject2,... The entries in the list of operations for triangles are the same as for *Triangle 3rd vertex*, except that the sides of the triangle are also displayed. The types of triangles in the list of operations, more precisely the triangle's 3rd vertices, can be set and changed by the user with a click on *Options/Triangle 3rd vertex*.

The command *Generic/Quadrilateral on segment* creates a rule that yields various types of positively oriented quadrilaterals with two given adjacent vertices. The rules for quadrilaterals are named Qobject1, Qobject2,...

The rules for triangles and quadrilaterals on a segment are ordinary rules and can be used in constructions just like other rules. However, they are primarily used to create the object at the very beginning of the construction process. When we study the properties of various triangles or quadrilaterals, it is useful to initiate the generic construction with 2 points and a rule for triangle or quadrilateral.

An example with detailed description of the use of rules for triangles and quadrilaterals on a segment can be found in Section 5.4.

### 5.3.5 Rules for triangle objects

It is possible to create rules for all kinds of triangle objects. The rules are generic versions of most of the commands for creating triangle objects in Sketch menu *Special*. The rules for creating ETC centres are named ETC1, ETC2,..., the rules for triangle lines can take the names Tline1, Tline2,..., for circles Tcircle1, Tcircle2,... and so on.

All rules in this group require exactly three points as vertices of a triangle, and possibly one or more as an argument (for example, the Cevian triangle of a point). All rules of this group are also obtained in a very similar fashion. We illustrate the process for the case of rule for triangle circles. We activate the command *Generic/Triangle objects/Triangle circles*. In the form that appears (Figure 68, left), we checkmark exactly three triangle vertices. We can use a previously declared rule or decide to declare a new rule. In the latter case, after clicking OK, a second form appears (Figure 68, right). In this form, we select the type of circles to be included in the rule and click OK.

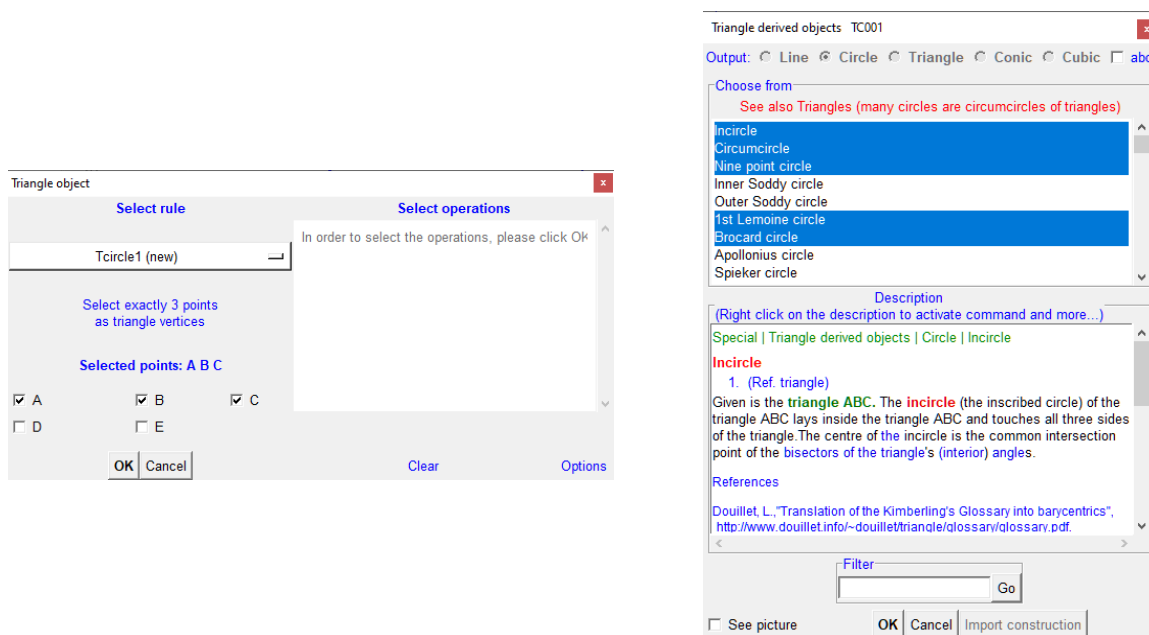


Figure 68

### 5.3.6 Rules for quadrilateral objects

It is also possible to create rules for all kinds of quadrilateral objects. The rules are generic versions of most of the commands for creating Quadrilateral objects in Sketch menu *Special*. The rules for creating centres of quadrilateral are named Qpoint1, Qpoint2,..., the rules for quadrilateral lines can take the names Qline1, Qline2,..., for circles Qcircle1, Qcircle2,... and so on.

All rules in this group require exactly four points as vertices of a quadrilateral. All rules in this group are obtained in a very similar fashion. We illustrate the process for the case of rule for quadrilateral centres (points). We activate the command *Generic/Quadrilateral objects/Quadrilateral points*. In the form that now appears (Figure 68, left), we checkmark exactly four vertices. We can use a previously declared rule or decide to declare a new rule. In the latter case a second form appears after clicking OK (Figure 69, right). In this form, we select the quadrilateral points to be included in the rule and click OK.

Quadrilateral objects are only partially analogous to triangle objects in generic constructions. Triangle objects are declared with three vertices, e.g., the Euler line of the triangle ABC, and are independent of the order in which the vertices are selected (the Euler line of the triangle ABC is the same as the Euler line of the triangle CBA). This is not the case with quadrilateral objects. A selected object, e.g. QL-Miquel point (QL-P1), on the quadrilateral with vertices A, B, C, D (in this order) will yield the QL-Miquel point of the quadrilateral ABCD only. The QL-Miquel points of the QA-siblings or QL-siblings or even QG-siblings of the quadrilateral ABCD may differ from that of the quadrilateral ABCD and they are not taken into account in the generic construction.

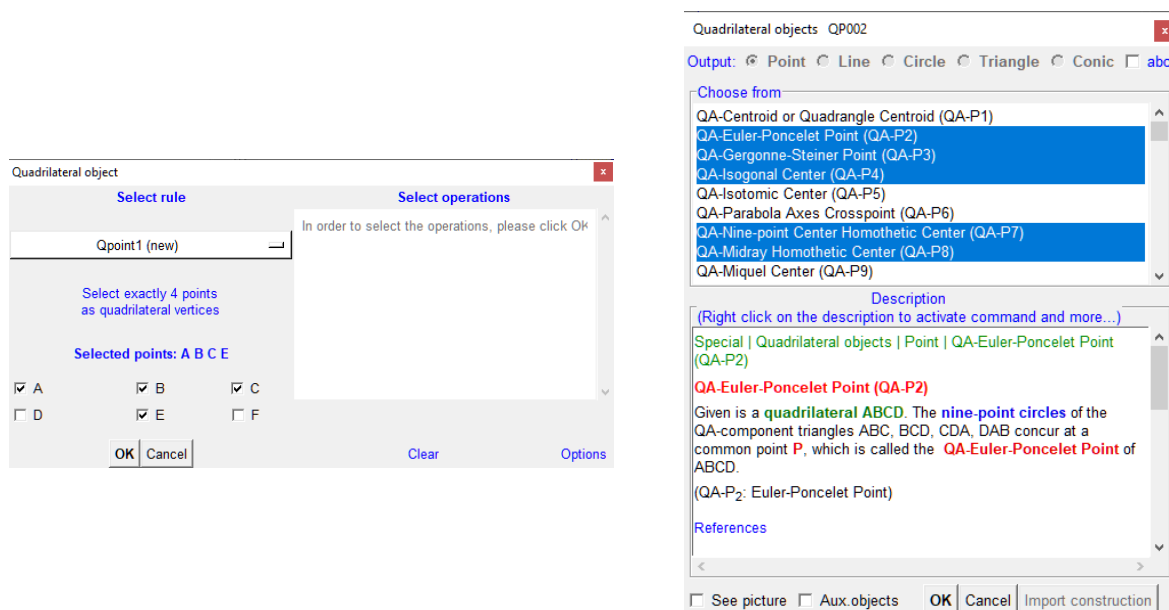


Figure 69

### 5.3.7 Editing rules

Creating a generic construction requires some preparation, since the user has limited control over the rule once it is declared. We present some notes on this here.

It is important to bear in mind that a generic construction cannot be modified when the Examples-form is displayed (see Section 5.2.2). The rules of a generic construction can only be changed when the Examples-form is not displayed.

The command *Generic/Manage rules* allows to do changes to the existing rules. It is possible to delete a rule, but note that deleting a rule does not delete the object(s) created by the rule in the base example. It is also possible to change the rule's selection of operations. Similarly deleting an object, created by a rule, does not delete the rule, only the object.

On the Examples-form you may have noticed a button **More...** in the upper right corner. This button that provides access to some commands.

The first command **Manage rules (locally)** allows you to manage rules – apparently is the same command as in the *Generic* menu. But there is one big difference. The command in the Examples-form only modifies the rules locally. When the Examples-form is closed, the rules are the same as they were when the Examples-form was opened. So it is possible to change the rules locally while working on examples without changing the generic construction.

The button **More...** also provides access to a command that is rarely used but can be very useful when needed. It is the **Set as base example** command. The command makes the current (displayed) example as the base example of the generic construction (and yes, this changes the generic construction). When the base example of a created generic construction is not appropriate (e.g. some constructed points coincide in the base example, but not in others), it is a good idea to use this command. The command should be used with caution, because all examples are obtained by applying rules to the base example.

The third command in the **More...** menu is **Set examples to Review**. At any time you can set the examples to be reviewed with the StartReview/Continue buttons.

## 5.4 Examples of generic constructions

### Example 1

Consider in a quadrilateral  $ABCD$  the incentres  $A'$ ,  $B'$ ,  $C'$ ,  $D'$  of the triangles  $BCD$ ,  $CDA$ ,  $DAB$ ,  $ABC$ . Is  $A'B'C'D'$  a cyclic quadrilateral (see Figure 70)? Is it cyclic for some special quadrilaterals? What if  $A'$ ,  $B'$ ,  $C'$ ,  $D'$  are other triangle centres?

A generic construction can help to find an answer to the above questions at an observation level. For this purpose, we create a generic construction that considers different types of quadrilaterals and different centres, e.g.,  $X_1$  to  $X_5$ .

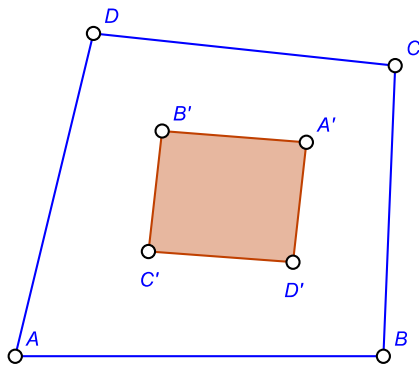


Figure 70

There are several ways of working out this task. We show here the one based on properties as criteria when scanning the examples. Here is a detailed description of the whole process (including the creation of the generic construction). The obtained generic construction is in file *Generic\_04.p*<sup>44</sup>.

1. We start by declaring a rule *Qobject1* for quadrilaterals.
  - Draw the points A and B.
  - Activate the command *Generic/Quadrilateral on segment*.
  - Fill the form: checkmark points A and B and choose the types of quadrilateral. In our case we selected the following quadrilaterals: random, bicentric, cyclic, diametric, and tangential.
  - Then click OK.
2. Let us declare the rule *Pobject1* for the triangle centres  $X_1$  to  $X_5$ .
  - Activate the command *Generic/GPoint*.
  - In the list of operations select *ETC triangle centre  $X_1$ , ... to ETC triangle centre  $X_5$* .
  - Checkmark the points B, C, D. (In this case, the order is irrelevant, as is FixFirst status.)

<sup>44</sup> OKExamples\OKG\_Plus\Generic\_04.p



- The rule *Pobject1 (new)* is declared after you press the button OK.
- The incentre of triangle BCD appears. Label it A'. (A' is thus the 'variable' point created by the rule *Pobject1* in triangle BCD.)
- 3. We apply the rule *Pobject1* to the triangles CDA, DAB, ABC.
  - Activate the *Generic/GPoint* command.
  - Select the rule *Pobject1*. A list of previously selected operations appears.
  - Checkmark the points C, D, A. (In this case the order is irrelevant, as is the FixFirst status.)
  - Press OK. The incentre of the triangle CDA appears. Label it B'.
  - Repeat these four steps for the triangles DAB and ABC to get points C' and D'.
- 4. The generic construction is ready. To start the observations, click on the **Gen** button on the toolbar on the right of the display.
  - In the form that appears, click on the **Manage properties** button. Select the red coloured **Shapes (all)** entry. This entry includes the observations of the shapes of triangles and quadrilaterals. Fill in the form as shown in Figure 71.
  - Click on the **Properties done** button (otherwise you will not be able to continue).

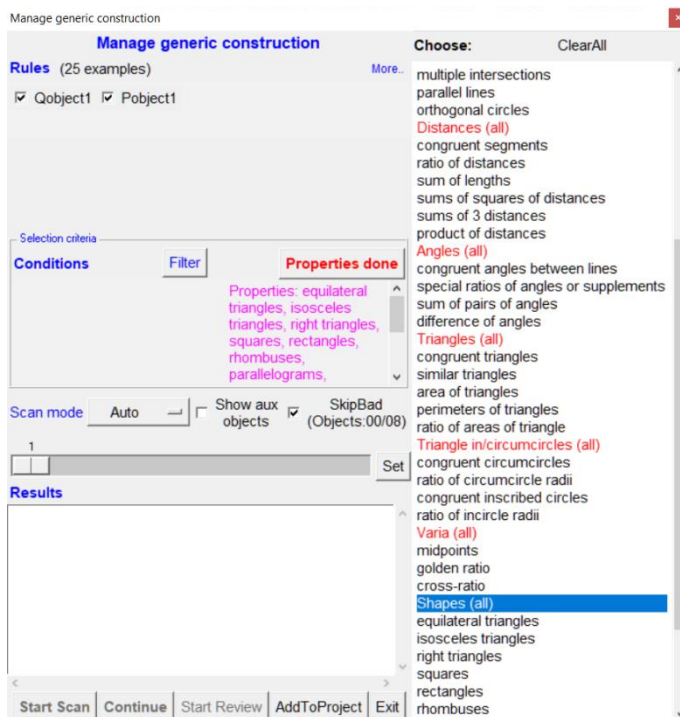


Figure 71

4. We are interested only in the shape of quadrilateral A'B'C'D', so it is important to restrict the observations to these points. Therefore, in the **Treatment** section of the observation pane fill in the entry **Consider only points** with A'B'C'D'.
5. Press **Start Scan** button. All examples will be scanned.
  - Press **Start Review** and then repeatedly **Continue** to review the results of observations. You can see the observed properties of each found example in the **Observed properties** pane.

## Example 2

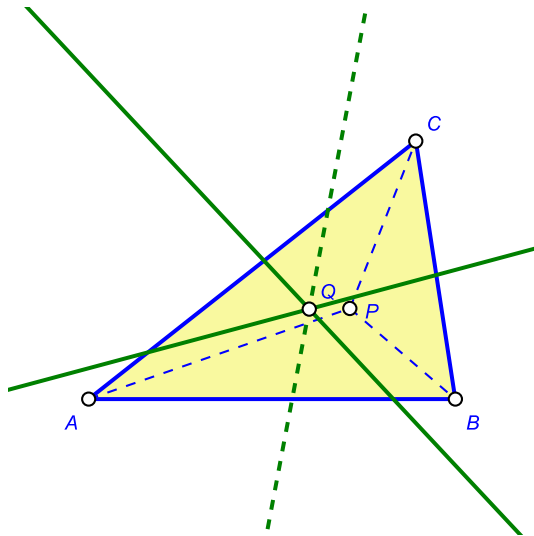


Figure 72

Let  $P$  be the incentre of a given triangle  $ABC$ . It is well known that the Euler lines of the triangles  $PBC$ ,  $PCA$ ,  $PAB$  concur in a point  $Q$ . Let us explore whether a similar concurrence occurs when  $P$  is a different triangle centre and when a different type of triangle lines is considered.

We present a brief description of how to create and analyse the generic construction for this investigation. You can download it as file *Generic\_05.p*<sup>45</sup>.

1. Construct a triangle  $ABC$ . (You can use the command *Special/A triangle*)
2. Use the *Generic/GPoint* command to declare the rule for the ETC centres  $X1$  to  $X5$  of the triangle  $ABC$ . On the displayed base example the point  $P$  (as incentre  $X1$ ) appears.
3. On the base example create the line segments  $PA$ ,  $PB$ ,  $PC$ .
4. Use the *Generic/Triangle objects/Triangle lines* command to create the rule  $Tline1$  for triangle lines of the triangle with vertices  $P, B, C$ . Select all available lines as operations. When you are done the Euler line of the triangle  $PBC$  appears.
5. Using the *Generic/Triangle objects/Triangle lines* command, apply the  $Tline1$  rule on the triangle  $PCA$  and then on  $PAB$ . The Euler lines of these triangles should appear on the screen.
6. Construct a point  $Q$  as the intersection of two of the constructed (Euler) lines. If the three (Euler) lines concur then  $Q$  is the point of concurrence.
7. Construct a condition *concur\_xxx* for concurrency of the three lines. Use the command *Advanced/Check property/ConcurrentL*.
8. Press the **Gen** button. The Examples-form appears. Checkmark both rules ( $Tpoint1$  and  $Tline1$ ). Checkmark also the *Concu\_xxx* condition. For Scan mode choose the option **Auto**.
9. Press the button **Start Scan**. The list of results (examples) is displayed. Press the **Start review** and then the **Continue** button to view and analyse each case in the result list.

<sup>45</sup> OKExamples\OKG\_Plus\Generic\_05.p

Scan started (60 examples)

Example 1 of 60

Pobject1: P = ETC triangle centre

x(1) of ABC

Tline1: L2 = Euler Line(PCA)

Tline1: L3 = Euler Line(PAB)

Tline1: L1 = Euler Line(PBC)

Concu\_xxx TRUE

Example 11 of 60

Pobject1: P = ETC triangle centre

x(1) of ABC

Tline1: L2 = Brocard axis(PCA)

Tline1: L3 = Brocard axis(PAB)

Tline1: L1 = Brocard axis(PBC)

Concu\_xxx TRUE

Example 37 of 60

Pobject1: P = ETC triangle centre

x(4) of ABC

Tline1: L2 = Euler Line(PCA)

Tline1: L3 = Euler Line(PAB)

Tline1: L1 = Euler Line(PBC)

Concu\_xxx TRUE

Example 42 of 60

Pobject1: P = ETC triangle centre

x(4) of ABC

Tline1: L2 = van Aubel Line(PCA)

Tline1: L3 = van Aubel Line(PAB)

Tline1: L1 = van Aubel Line(PBC)

Concu\_xxx TRUE

Example 47 of 60

Pobject1: P = ETC triangle centre

x(4) of ABC

Tline1: L2 = Brocard axis(PCA)

Tline1: L3 = Brocard axis(PAB)

Tline1: L1 = Brocard axis(PBC)

Concu\_xxx TRUE

Done

### Example 3

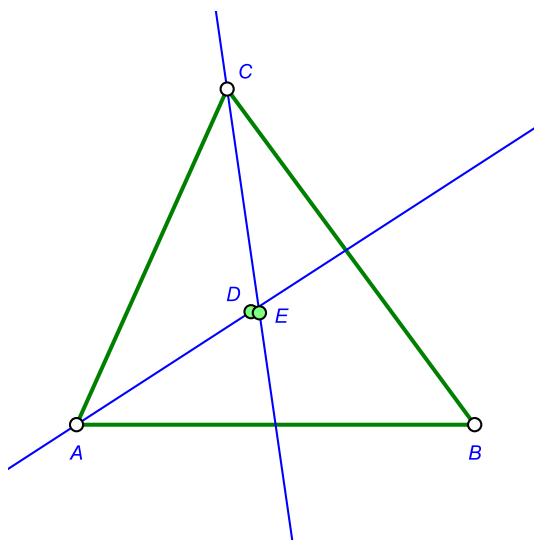


Figure 73

Let D and E be two ETC centres of a given triangle ABC (Figure 73). Does it happen for some pair of centres that the lines AD and CE are perpendicular? If not for general triangles, does it happen for some special type of triangles?

To investigate this we make a generic construction<sup>46</sup> (file Generic\_06.p). We shall consider the centres X1 to X29 for D en E. We shall consider the following types of triangles: a random triangle, a triangle with angle C=60 degrees, a triangle proportional to the 6,9,13 triangle, a triangle where  $a=2b$ .

Here is an overview of the construction steps:

1. Draw the points A and B.
2. Use the *Generic/Triangle on segment* command to create the rule for a triangle with A, B as two vertices. In the form choose the following 3rd vertices operations: random, C=60,  $a:b:c=6:9:13$ ,  $a:b=2$ .

<sup>46</sup> OKExamples\OKG\_Plus\Generic\_06.p

3. Use the command *Generic/Triangle objects/More ETC centres* to create the rule ETC1 for point D. Checkmark the points A,B,C and select the centres X1 to X30. The point D appears as the X1 centre of ABC.
4. Use the *Generic/Triangle objects/More ETC centres* command to create the ETC2 rule for point E. Checkmark points A,B,C and select the centres X2 to X30. The point E appears as X2 centre of ABC. Note: Two separate rules are needed for the points D and E. The selected operations start from X2 to avoid overlapping of points D and E in the base example.
5. Construct lines AD and CE. Use the *Advanced/Check property/PerpendicularL* command to create the condition parameter *Perpe\_XX* for the perpendicularity of AD and CE.
6. The generic construction is ready. It contains almost 3500 examples. To check the condition of perpendicularity on them, press the button **Gen**.
7. In the Examples-form checkmark all the rules as well as the *Perpe\_xx* condition. For scanning mode choose the **Fast** option. OK Geometry observes that 3 examples satisfy the condition.
8. Use **Start Review** and **Continue** to identify them.

## 5.5 Triangle analysis of generic constructions

A generic construction contains some or many examples of constructions. Is it possible to do triangle analysis for all examples of a generic construction? Yes, it is possible, in many ways. We illustrate this with the example from Section 4.1. In Figure 74, the point P is the orthocentre of the triangle ABC. The points A', B', C' are the centres of the circles passing through the points P,B,C, the points P,C,A, and the points P,A,B. We have observed that the lines AA', BB', CC' concur in a point (the related check of the property showed *concur\_xxx=True*). We wonder if the point of concurrency Q of the three lines is a known triangle centre.

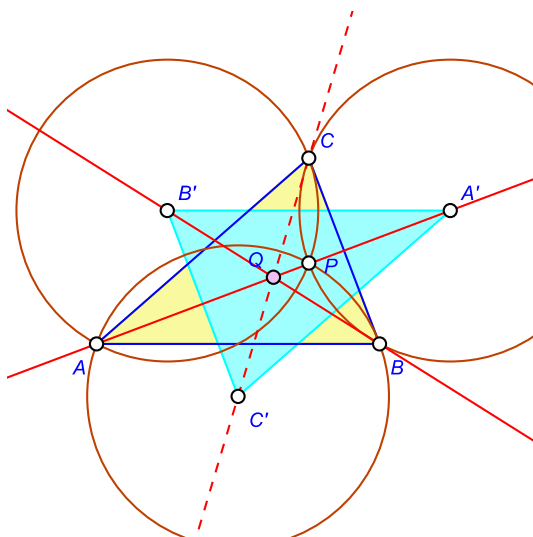


Figure 74

The OK Geometry file *Generic\_03.p*<sup>47</sup> is a generic version of the construction described. It consists of two rules: Pobject1 for the point P yields the ETC centres X1 to X5 of the triangle ABC, and the rule Cobject1 yields the circumcircles, the incircles, and the 9-point circles of triangles PBC, PCA, PAB. To the construction already mentioned<sup>48</sup> (file *Generic\_02.p*) we added the point Q as the intersection of the lines AA' and BB'. In examples where the lines AA', BB', CC' concur, Q is the point of concurrency<sup>49</sup>. We will study the position of point of concurrency Q with regard to the triangle ABC in all such examples of the generic construction.

We first show a simple way to find the position of the point of concurrency in examples where the lines AA', BB', CC' concur.

1. Activate the command **Manage generic construction** (just click on the **Gen** button in the toolbar on the right side).
2. In the form that appears checkmark the rules Pobject1 and Cobject1.  
Also checkmark the *Concu\_xxx* condition, as we will only consider examples where concurrency occurs.  
For the Scan mode choose the **Criteria** option.
3. Press **Start Scan**. The first example in which the concurrency occurs is displayed.
4. If you want to examine the position of point Q in the reference triangle ABC, simply activate the **Analyse object wrt. triangle** command (or press the corresponding button on the toolbar in the right) and pick the point Q. Figure 75 shows the output in one of the examples of generic construction (note that we set the number of centres considered to about 16000 and used the **More** option).
5. Click on the **Continue** button for the next example that satisfies the criteria.
6. Repeat the last two steps as needed.

---

<sup>47</sup> OKExamples\OKG\_Plus\Generic\_03.p

<sup>48</sup> OKExamples\OKG\_Plus\Generic\_02.p

<sup>49</sup> OKExamples\OKG\_Plus\Generic\_03.p

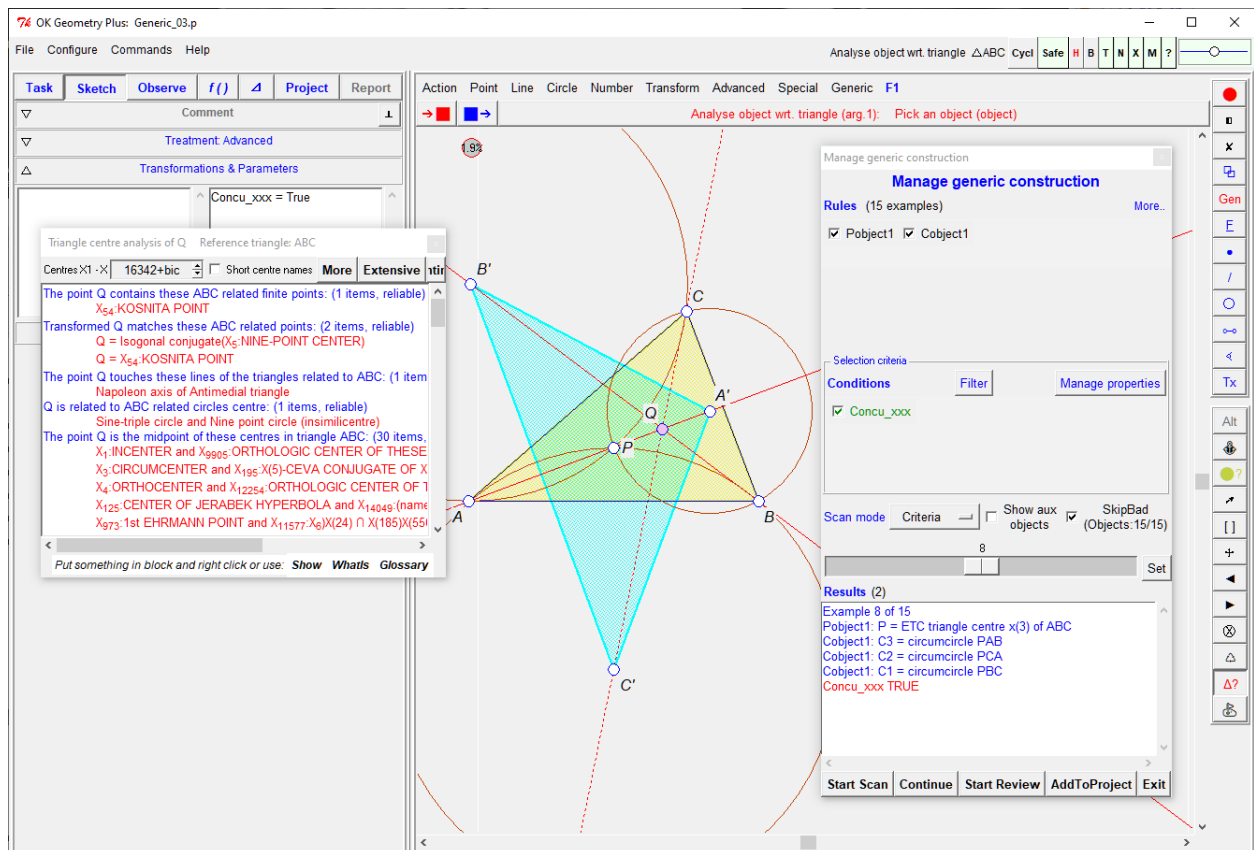


Figure 75

The above method is time-consuming when there are several examples to be considered in the generic construction. In such cases is better to use a faster automatic method as follows:

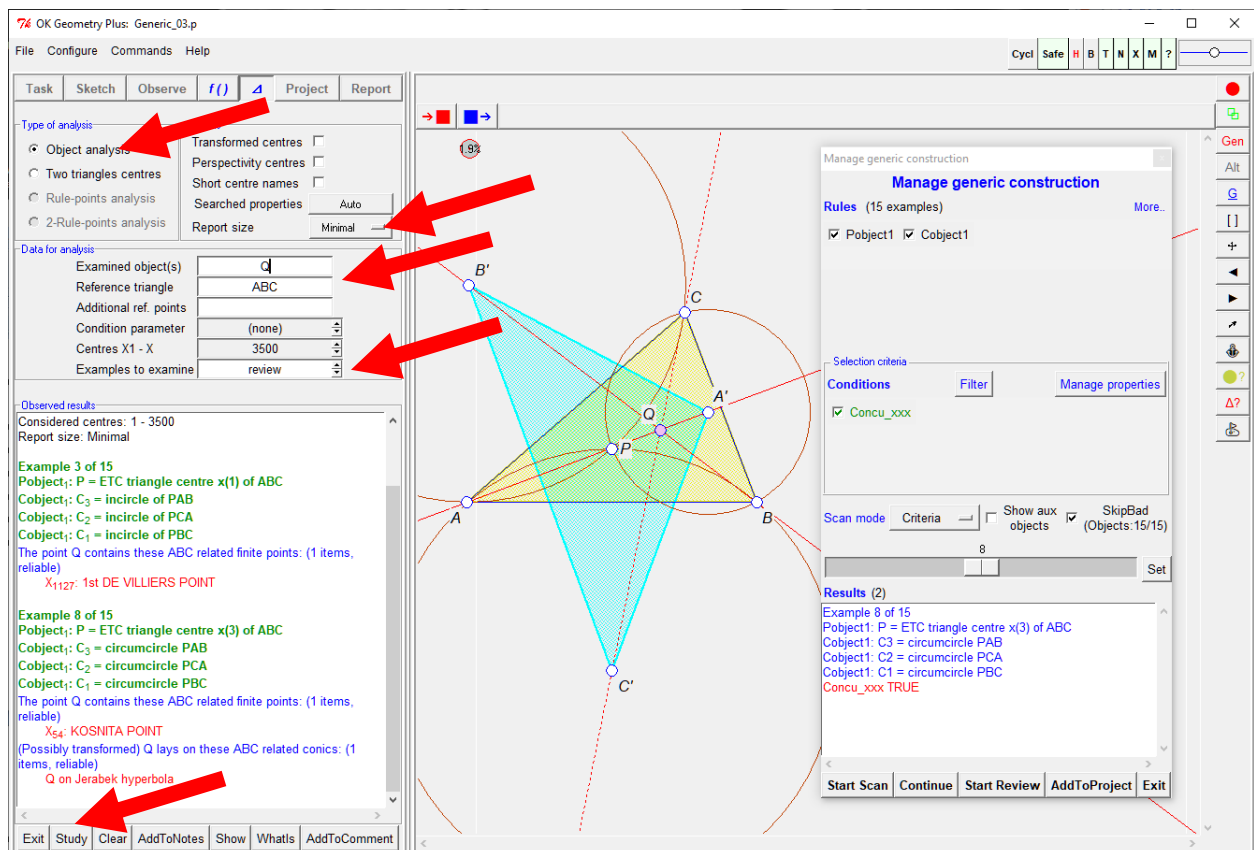


Figure 76

1. Activate the **Manage generic construction** command (just click the **Gen** button on the toolbar to the right).
2. In the form that appears, checkmark the Pobject1 and Cobject1 rules.  
Also checkmark the *Concu\_xxx* condition, since we will only consider examples where concurrency occurs.  
For the Scan mode choose the **Fast** option.
3. Press **Start Scan** and wait a bit. A list of the sequential number of examples that satisfy the condition of concurrency appears. (You can at any time, now or later, display and analyse these examples, but for now we will not.)
4. Activate the **Triangle analysis** module (the button with the triangle in the main menu in the upper left corner of the display).
5. Carefully fill some data in the form that appears (Figure 76):  
Select **Object analysis** (it is set by default).
6. Set **Report size** to **Minimal**. Later you can repeat the analysis with other option. It is advisable to start with the **Minimal** option, otherwise the results can take up a lot of space and time.  
In the **Examinated object(s)** entry write the object under analysis (in our case Q).  
In the entry **Reference triangle** write ABC, our reference triangle.  
Set the range of **Considered triangle centres** according to your wishes.
7. The most important thing is the entry **Examples to examine**. We set it to **review** since we want to do the analysis only for examples in the selection of examples that satisfied the criteria. The

option **scan** examines all examples, while the option **current**, obviously, examines the current example only.

8. Click on the **Study** button at the bottom menu and wait for the results.

The Triangle analysis offers many other options that can be profitably used together with the **Manage generic construction** form (i.e. the Examples-form).

## 5.6 Formulae observation in generic constructions

A generic construction is a family of constructionally isomorphic dynamic constructions called examples of a generic construction. All examples of a generic construction are built of the same objects, though they are not related in the same way, so the properties of the examples are not the same. Due to constructional isomorphism of the examples of a generic construction, it is possible to observe the relationships between geometric quantities for a particular or for all the examples of a generic construction.

All three variants (**simple**, **advanced**, **triangle**) of **Observing formulae** (see Section 4) can be applied to generic constructions in the very same way as on ordinary dynamic constructions. As usually, we explain the working procedure with a simple example:

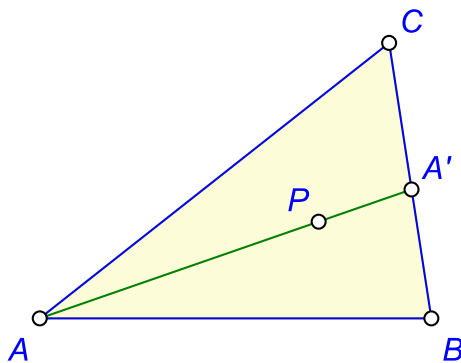


Figure 77

It is well known that in any triangle the centroid (X2) divides each median of the triangle in the ratio 2:1. Let us examine the analogous ratios for some other triangle centres: X1 (incentre), X2 (centroid),...X11 (Feuerbach point). We proceed as follows:

### Step 1. Provide the generic construction to be investigated.

For convenience, a generic construction is provided in the file *Generic\_07.p* (Figure 77), so that you can open this file in OK Geometry<sup>50</sup>. In the generic construction the position of the point P is

<sup>50</sup> OKExamples\OKG\_Plus\Generic\_07.p



subject to the rule Pobject1. The generic construction consists of 11 examples, where the point P takes the place of centres X1, X2, ..., X11 of the triangle ABC.

**Step 2. Activate the generic construction with the **Gen** button. Select the rules to apply.**

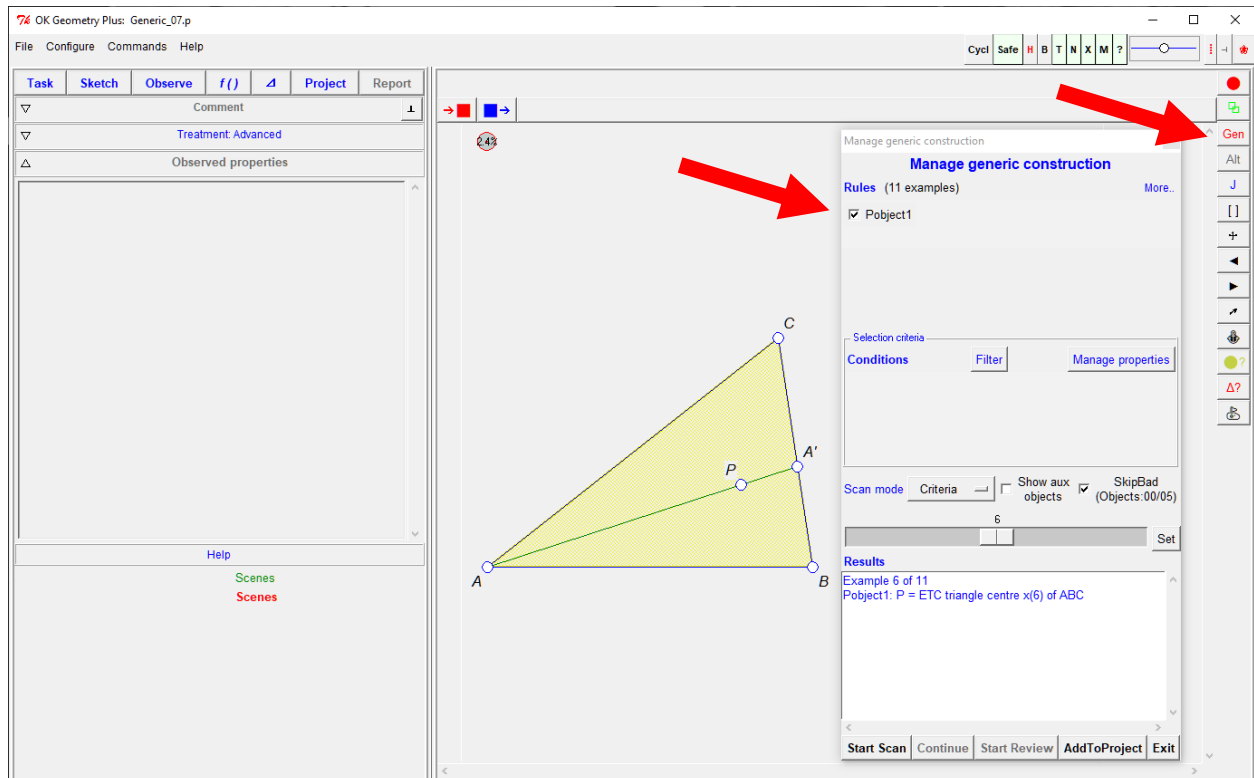


Figure 78

The **Gen** button activates the form for managing examples of generic construction (Figure 78). In the form checkmark the Pobject1 rule.

**Step 3. Invoke the command *Commands/Observe formulae/Triangle*.**

The form for **Observe formulae** will appear (Figure 79).

You can as well call the **simple** or **advanced** variant of the **Observe formulae** command and proceed in an analogously as described below.

**Step 4a. Observing a specific example of a generic construction.**

1. Set the slide bar in the examples manager to the desired value (Figure 79). For example, if you want to consider the example 6 (i.e. P is the symmedian point of ABC), set the slide bar to 6.
2. Click the Set button (next to the slide bar). The example 6 appears.
3. Fill the **Observe formulae** form, in our case we observe the ratio  $d(AP):d(PA')$ . In particular, be sure that the entry **Examples** is set to **current**.
4. Click on the Observe button in the **Observe formulae** form. The results appear in the results area. You can use the results as described in Section 4.2.3.

Repeat steps 1 to 4 for additional examples as desired.

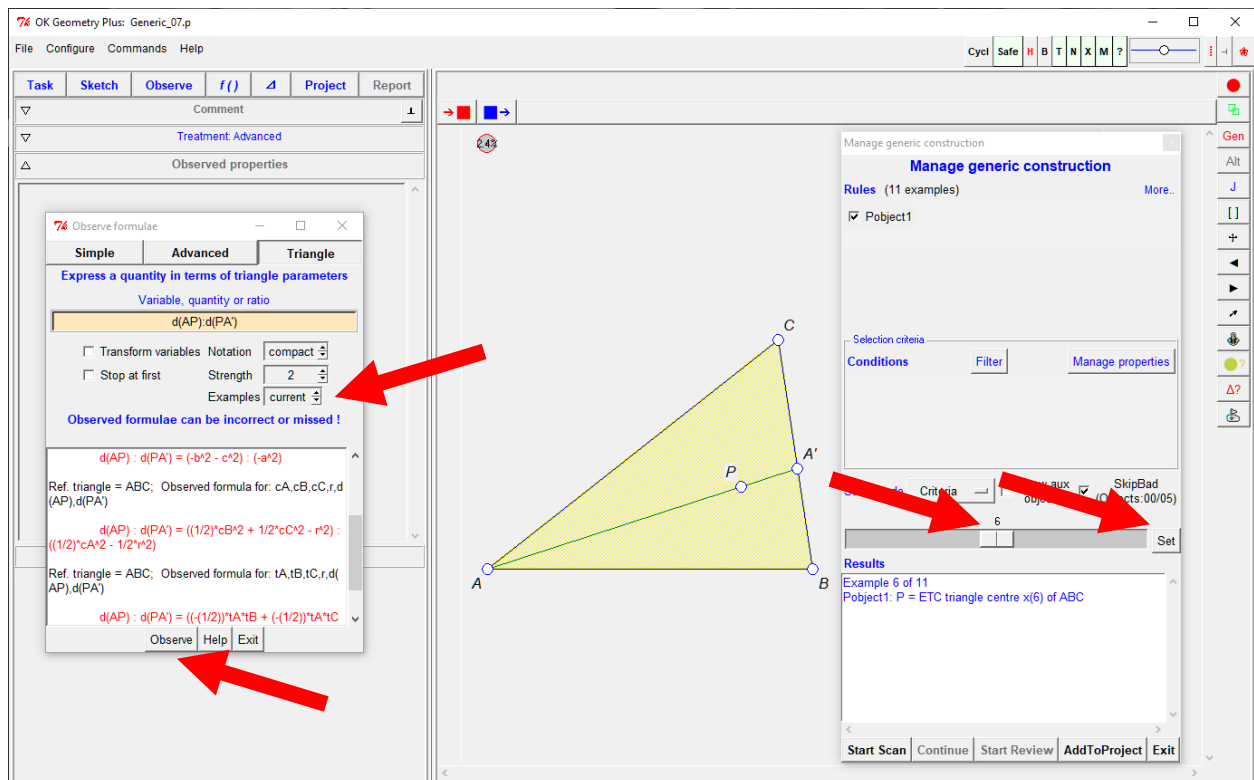


Figure 79

#### Step 4b. Automatic observation of all examples of a generic construction.

1. Fill the **Observe formulae** form, in our case we observe the ratio  $d(AP):d(PA')$ . In particular, be sure that the entry **Examples** is set to **scan** (Figure 80).  
Note that the **triangle** variant of **Observe formulae** command for all examples can be very time consuming, especially if **Strength** is set to 3 or 4 and/or **Transform variables** is checkmarked. Checkmarking **Stop at first** cuts the computation time, but you obtain only one (not necessarily the most elegant) formula for each example of the generic construction.
2. Click on the **Observe** button in the **Observe formulae** form. The list of results for all examples appears in the results area. You can use the results as described in Section 4.2.3. You can stop the observation by pressing the Escape key on the keyboard.
3. At the end of the observation, you are asked whether to pass to the examples manager the information for which examples something useful was found. If you choose to do so, you can review the examples with the **Begin Review** and the **Continue** buttons in the form for managing examples (Figure 80).

For the considered generic construction OK Geometry has suggested various expressions for the ratio  $d(AP):d(PA')$  for all examples, i.e. for all centres  $X_1$  to  $X_{11}$ .

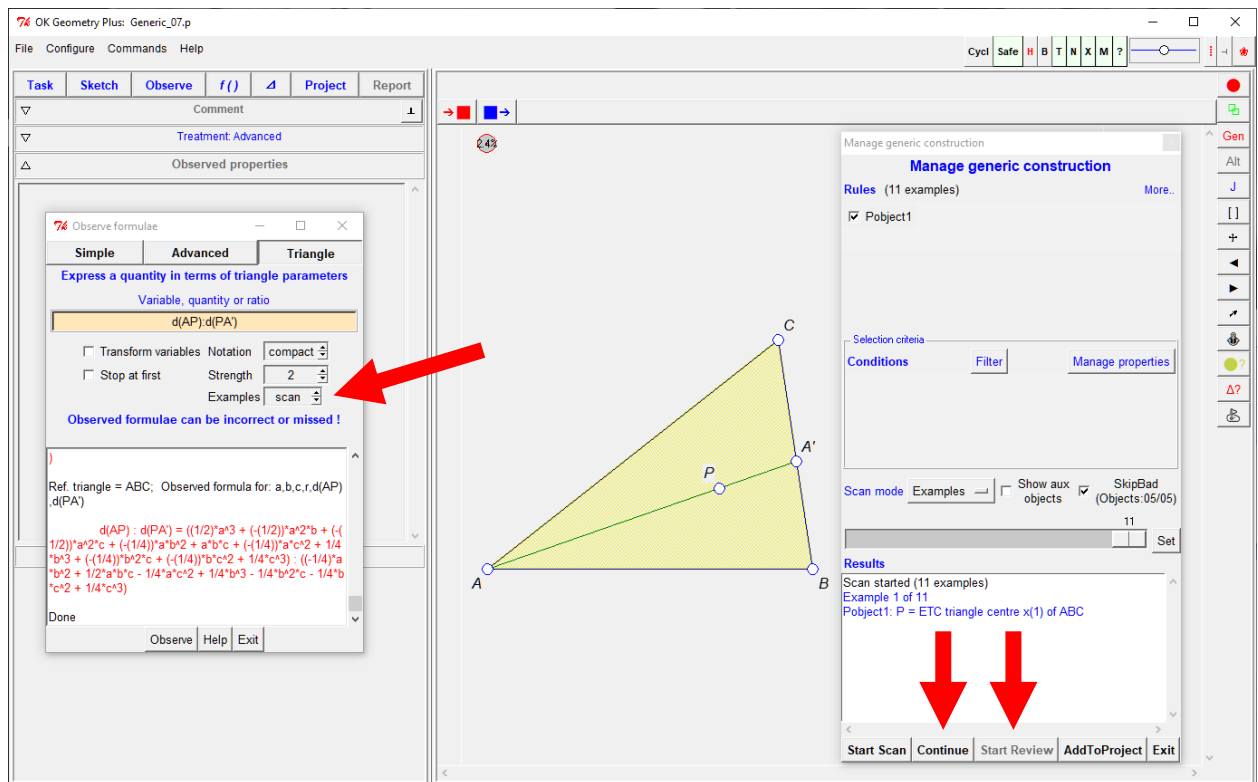



Figure 80

## 6 Using the Working archive of constructions

During the work, the constructions created in OKG can be automatically and deliberately archived in a working archive. You declare your working archive file in the *Configure/General options/Archives* form. When you wish (e.g., if the archive becomes too large), set a new archive file. If no archive file is specified, constructions are not archived.

Each time you use the command *New construction/New Project* or you exit the program, the **current construction** (not the current project!) is archived. You can also archive the current state or variant of a

construction while you work on it by clicking on the *Snapshot for Working archive button*  in the toolbox on the right-hand side of the display. In this case, you can add a tag and comments to the archived construction.

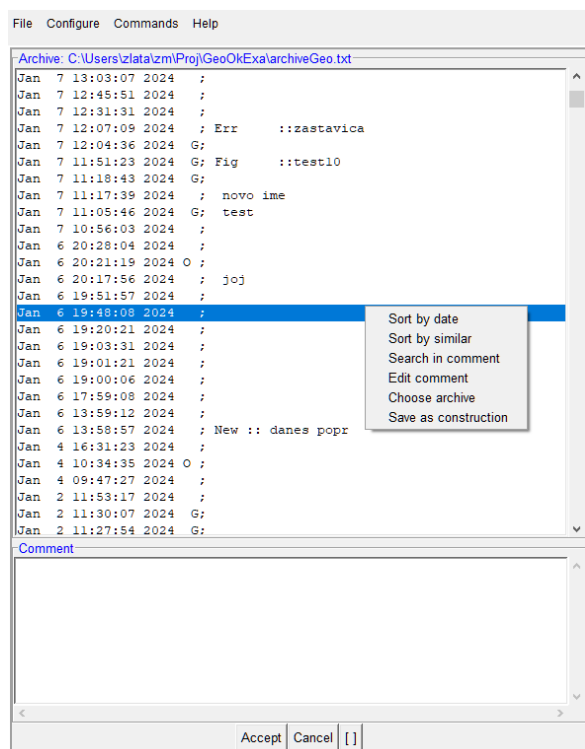


Figure 81

You can review and restore the archived constructions with the command *File/Inspect working archive* (Figure 81). You can see the date and basic comment for each archived file. Furthermore, a G and an O indicator before the semicolon show that the construction is a generic construction (G) or it contains an optimisation procedure (O). You can perform various operations on the archived files by right-clicking on them. In the archive you can, for example, find previous geometrically similar constructions to the current construction or to a selected archived construction. With the **Accept** button, the selected archived construction becomes the current construction.

The archive is useful for restoring constructions from past work and for quickly saving ideas of any kind that arise during work.

## 7 Using the GeoGen archives of constructions

### 7.1 Introduction

GeoGen is a software programme developed by **Patrik Bak**. Its aim is to generate non-trivial proving problems in planar geometry, i.e. planar geometric configurations together with associated properties that are not easy to prove. You can access the complete development version of GeoGen on GitHub<sup>51</sup>. If you are not familiar with development versions, you can find an easy-to-install ‘publish’ version of the parts of GeoGen that are called up from OK Geometry on the download page of OK Geometry. To run GeoGen from OK Geometry, unzip the files from GeoGenPublish.zip into a directory of your choice. Both installationions of GeoGen require a .NET 7.0 runtime environment<sup>52</sup>.

To use GeoGen in combination with OK Geometry you need to specify in the OK Geometry configuration (*Configure/General options/Archives*) the directory that contains the GeoGen execution file(s) and the directory of your working files related to GeoGen.

Before we explain the concept on not-trivial tasks, we will introduce the workflow with GeoGen. The process of creating non-trivial proving problems (and the associated configurations) consists of three steps:

1. Designing the family of considered configurations	<p>You can perform this phase within OK Geometry with the command <i>File/Generate GeoGen archive</i>.</p> <p>The resulting designs are normally saved in files with prefix ‘input_’ and extension ‘.txt’, e.g. <i>input_test.txt</i> .</p>
2. Generating the set of non-trivial problems	<p>For this <b>essential phase</b>, OK Geometry calls the programme GeoGen by Patrik Bak as an external programme. GeoGen is executed separately and independently of OK Geometry. You start the generation from one or more input design files at once with the ‘<i>CreateGeoGen archive</i>’ button. The generation process can take from a few seconds to several hours, depending on the complexity and the number of configurations taken into account. However, you can stop the process at any time by simply closing the window in which the generation is running. As a result, you get the corresponding output files (archives) with prefix ‘output_’ and the extension ‘.json’ or ‘.txt’. For example, from the files <i>input_test1.txt</i>, <i>input_test2.txt</i>,... you get the files <i>output_test1.json</i>, <i>output_test2.json</i>,...</p>

<sup>51</sup> <https://github.com/PatrikBak/GeoGen>

<sup>52</sup> <https://dotnet.microsoft.com/en-us/download/dotnet/7.0>

3. Inspecting the generated set of problems	This phase can be carried out within OK Geometry using the command <i>Files/Inspect GeoGen archives</i> . You can inspect one or more output files (archives) at once. Each task in the archives can be visualised, set as an icon in the current project, saved as a OK Geometry construction, transferred to the working archive. An inspected task can also be transferred to OK Geometry as the current construction so that it can be analysed immediately with the tools available in OK Geometry.
---	--

GeoGen's 'reasoning' is based on variant of the deductive database method (GDD) method of Chou, Gao and Zhang. Important elements of this method are the list of **allowed operations** (construction steps) and the list of **inference rules** (generally known facts about properties).

Three steps are required to create the design for the family of considered configurations:

1. Start from the **initial configuration** (a generic triangle, a generic quadrilateral or a previously saved design to be possibly modified).
2. Create the **base configuration** by adding new objects by applying allowed operations to previously defined objects.
3. Select a subset of the allowed operations that will be used in all meaningful combinations (of objects and operations) in **further construction steps**. Here it is also necessary to specify the number of steps and some other parameters. Depending on the number of objects in the base configuration and the number of operations selected and the number of subsequent steps, the choices can easily lead to excessively large number of configurations to be considered. Moderation and good intuition are therefore required here.

GeoGen takes several steps to identify configurations with interesting properties from among those under consideration. First, irrelevant configurations are sorted out (e.g., only geometrically non-isomorphic configurations are considered). Then, GeoGen identifies some basic properties of the considered configuration by numerical observation (e.g. it finds that three points are collinear). The crucial step is to find out whether the observed property is of interest or not. To this end, GeoGen attempts to prove the property using a time-efficient, specially developed GDD method, i.e., to check whether the observed property can be deduced (in a given time interval) by applying inference rules from the constructional properties of the considered configuration. If this check fails, the property is declared as **non-trivial**. A non-trivial property can be (very very unlikely) the result of an observation error or (very likely) be true but the property cannot be inferred from the constructional properties of configuration using the available inference rules – at least not in the required time interval.

The observed properties of the base configuration are assumed to be true – and never treated as a problem. A property is considered a potentially non-trivial property only if it is dependent on all objects of the configuration by construction.

Finally, the author, Patrik Bak, provides criteria for evaluating the 'complexity' and 'beauty' of the found properties of configurations, according to his expertise in solving and designing geometry problems.

GeoGen only considers the following properties of configurations:

- collineation of three points,
- concurrency of three lines,
- concylicity of four points,
- congruence of two segments,
- tangentiality of lines to circles,
- parallelism of lines,
- perpendicularity of lines,
- tangentiality of circles,
- incidence of points and circles/lines.

GeoGen assigns to non-trivial problems respective relevance factors. In the OK Geometry display, the relevance ranges from 10 000 upwards. The higher the relevance, the more “difficult and beautiful” the problem is.

You can ask OK Geometry to also display the so-called simple tasks, i.e. the properties of configurations that GeoGen was able to show in a very short time (see Section 7.5). The simple problems have relevance 9200 – 10000. In general, although not as a rule, a higher relevance indicates a more difficult problem.

The details of the whole process can be found in the Patrik Bak master’s thesis<sup>53</sup>.

## 7.2 Using GeoGen from OK Geometry

We explain the use of GeoGen from OK Geometry with an example<sup>54</sup>. Consider a simple base configuration (Figure 82):

In a given triangle ABC, let D be the midpoint of BC. Let the bisector I of AD intersect the sideline AB in E and the sideline AC in F.

---

<sup>53</sup> [https://drive.google.com/file/d/1dsaxDCMzIAPfB3e4rd8ut2RuZ\\_sn2Zm5/view](https://drive.google.com/file/d/1dsaxDCMzIAPfB3e4rd8ut2RuZ_sn2Zm5/view)

<sup>54</sup> OkExamples\GeoGen\_examples\input\_mydesign.txt  
OkExamples\GeoGen\_examples\output\_mydesign.json

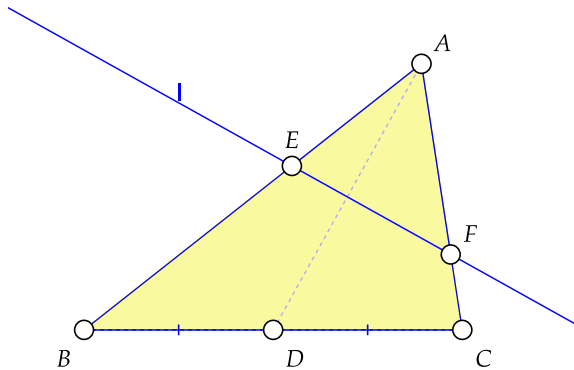


Figure 82

To create new configurations from this rather simple configuration, we add two new objects by applying two out of these three operations:

- a new point is the midpoint of two existing points,
- a new point is the circumcentre of the triangle formed by three existing points,
- a new point is the antipode of an existing point in the circumcircle of a triangle formed by that point and two other existing points.

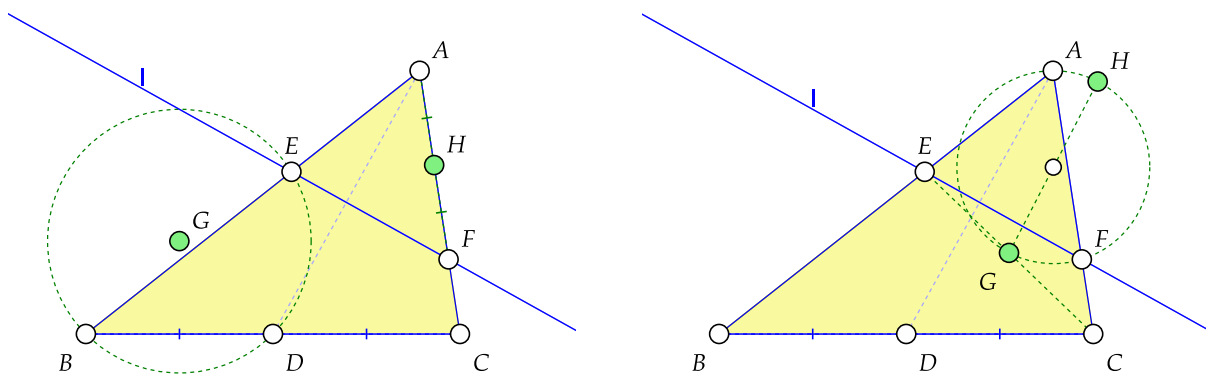

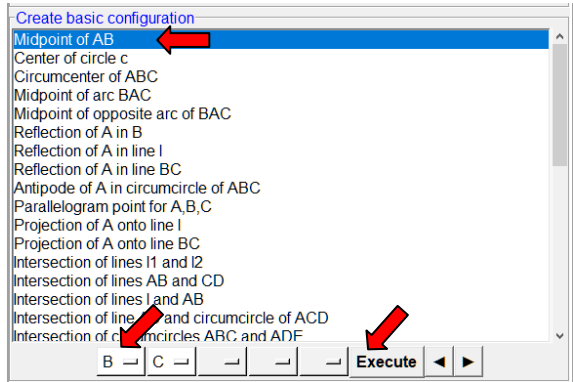
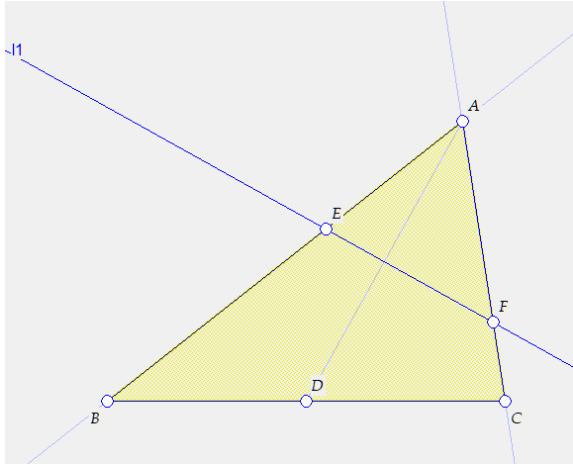


Figure 83

Figure 83 shows only 2 examples (out of 15295 possible cases, including the degenerate ones, but GeoGen reduces the family to about 4000 significantly different cases) of configurations obtained by adding the points G and H to the base configuration.

To create the design for the family of configurations described above, proceed as follows:



<p>Select the command <i>File/Generate GeoGen archive</i> in the main menu.</p> <p>A form dedicated for this purpose will appear.</p> <p>Proceed 'from top to bottom' In the form.</p>	
<p>Select the initial configuration, in our case it is a triangle.</p> <p>A triangle ABC appears on the right-hand side.</p>	
<p>Click on the operation 'Midpoint of AB'.</p> <p>Select points B and C.</p> <p>Then click on <b>Execute</b>.</p> <p>The midpoint D appears on the right-hand side.</p>	
<p>Proceed in the same way as above for the commands:</p> <ul style="list-style-type: none"> <li>• <i>Perpendicular bisector of AB</i> for the points A and D to obtain line l1,</li> <li>• <i>Intersection of lines l and AB</i> for the line l1 and the points A, B to obtain point E,</li> <li>• <i>Intersection of lines l and AB</i> for the line l1 and the points A, C to obtain point F.</li> </ul> <p>The base configuration is now complete. Note that the resulting figure on the right also contains auxiliary objects for easier interpretation. The auxiliary lines are bleached, you can make them more or less bleached with Alt+MouseScroll.</p>	

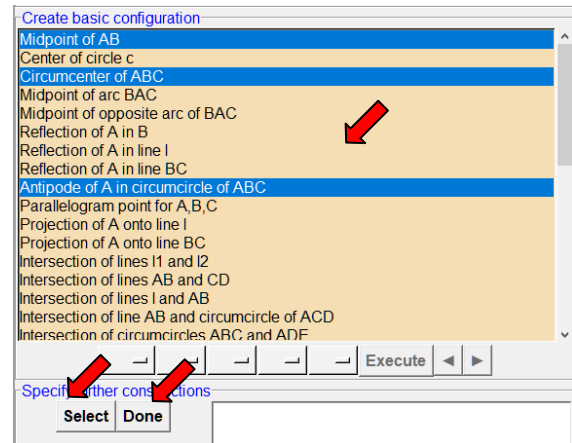
We proceed to select the list of further operations:

Click on **Select**. The list of constructions will now be coloured.

Select the operations to be included in the list.

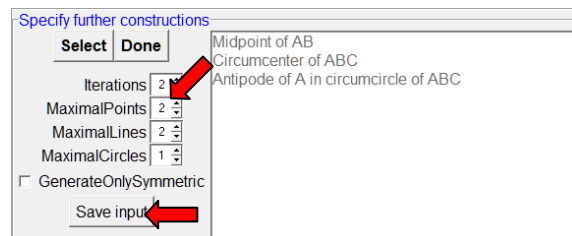
Then click on **Done**.

The selected operations are now displayed in a separate list.



Now set the parameters for further operations. In our case, we want 2 additional operations (2 *Iterations*) that will generate 2 points. The entries *MaximalLines* and *MaximalCircles* are irrelevant in our case.

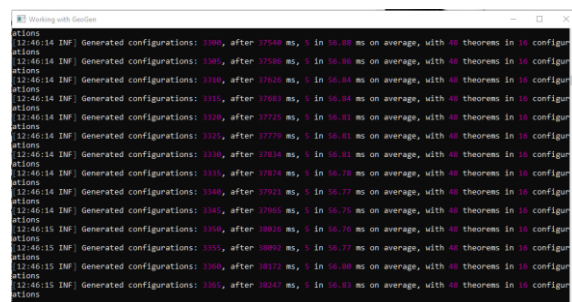
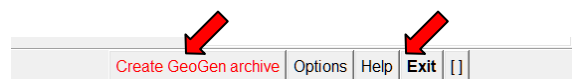
**It is important that you save your design as input file, otherwise you will not be able to generate the GeoGen archive of non-trivial problems.** If you write 'mydesign' as name of file, the design will be saved as file named 'input\_mydesign.txt'.



Once the designed has been saved, click on **Create GeoGen archive**. You can select one or more input files to be processed simultaneously. The command calls up GeoGen as external programme that is executed in a separate window. The execution may take some time, but you can stop it by simply closing the window. In the meantime you can do other things in OK Geometry.

For each input file, e.g. *input\_mydesign.txt*, GeoGen generates a corresponding output file, e.g. *output\_mydesign.json*. The output files are always located in the directory for GeoGen working files specified in the configuration.

To end the design of the GeoGen input files, click on **Exit** in the design form.



You can create several design files to be processed later or to be used as a starting point for new design files.

### 7.3 Inspecting GeoGen archives

The archive files created with GeoGen can be viewed in OK Geometry with the command *Files/Inspect GeoGen archives*. You can select several GeoGen archive files to be inspected at once.

Figure 84 shows what happens after we select the file *output\_mydesign.json*. GeoGen has found 58 non-trivial tasks related to the designed configurations. The tasks and configurations can be visualised, for each task you can also see the relevance, the symmetry and the complexity estimates, and the number of objects. In the visualisation you can also see auxiliary objects that are bleached, so that they are more or less visible, as well as the emphasised objects that illustrate the property found.

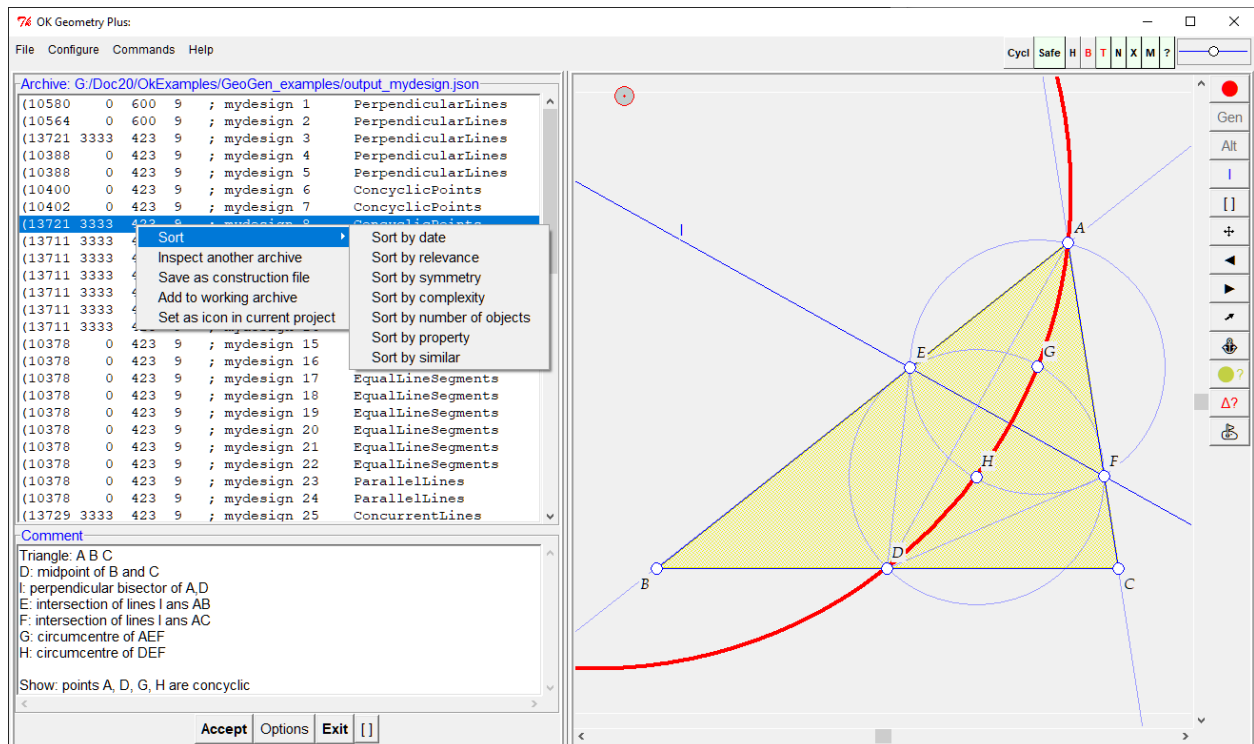


Figure 84

As you can see, the tasks can be sorted according to various criteria. The configurations and properties can

- be saved as ordinary OK Geometry constructions,
- be set as icons in the current OK Geometry project,
- be added to the current working archive.

The **Accept** button makes the visualised configuration the current construction in OK Geometry – so that you can analyse it immediately with the tools provided by OK Geometry. To exit without affecting the current OK Geometry construction, press **Exit**.

We present two examples that we have selected from the GeoGen archive (Figure 85). Both tasks are indeed beautiful and certainly not easy to prove. For the example on the right, the advanced observation with OK Geometry establishes that the circumcentre K of the triangle ABD lays on the circumcircle of the triangle BDE (and obviously on the bisector l). In fact, the point K is a path to the solution, but there is still some work to be done.

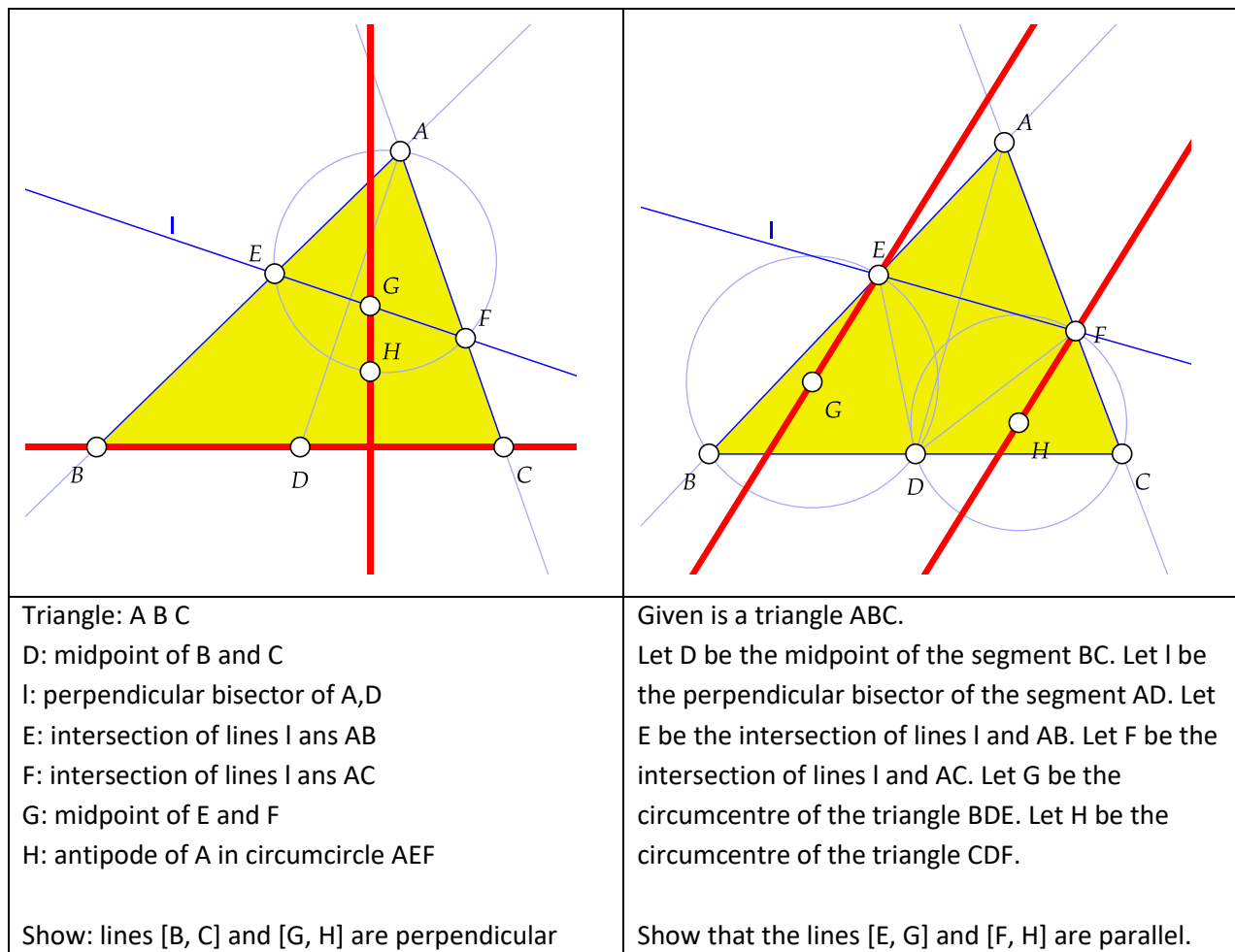


Figure 85

## 7.4 A second example

In the second example we present, we can see how OK Geometry can shed additional light and make more sense of a configuration of non-trivial tasks created by GeoGen. Figure 86 shows one of the tasks created from a rather simple GeoGen design input file<sup>55</sup>.

<sup>55</sup> OkExamples\GeoGen\_examples\input\_incenter.txt  
OkExamples\GeoGen\_examples\output\_incenter.json

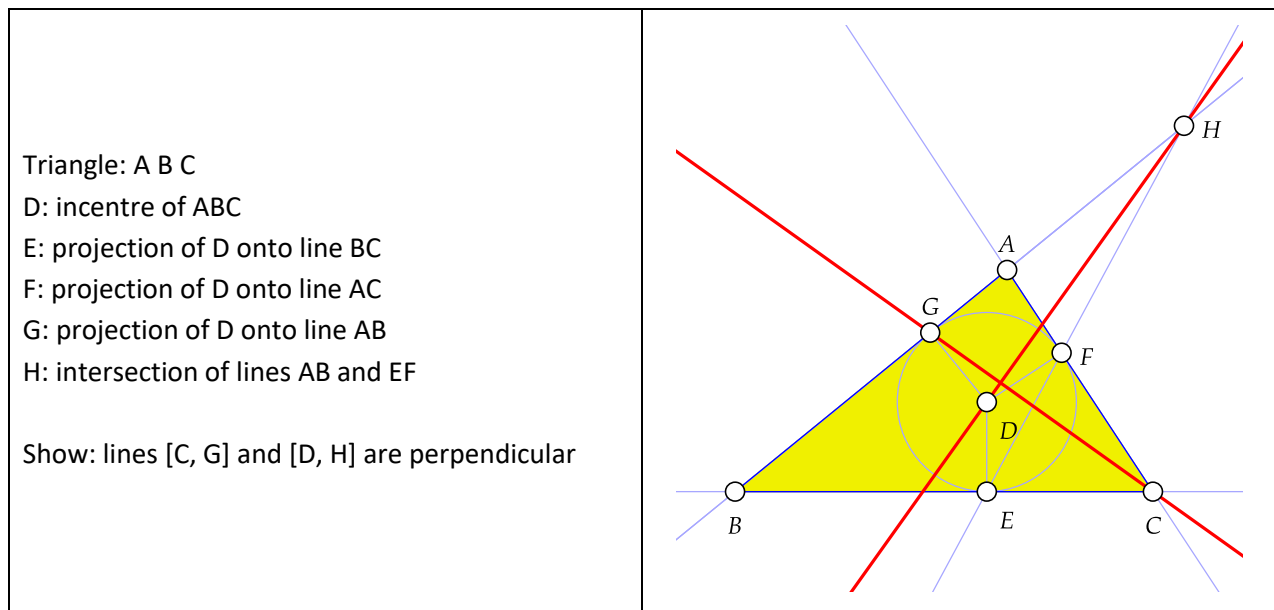


Figure 86

In order to better understand the situation (not necessarily for proof purposes), we make the configuration of the task the current OK Geometry construction. Obviously, a point I is created at the intersection of the lines CG and DH. We analyse the point I with respect to the triangle ABC – observation shows that I lays on the Brocard circle of the intouch triangle of ABC, i.e. on the Brocard circle of triangle EFG. We therefore analyse the point I with respect to the triangle EFG as a reference triangle (Figure 87).

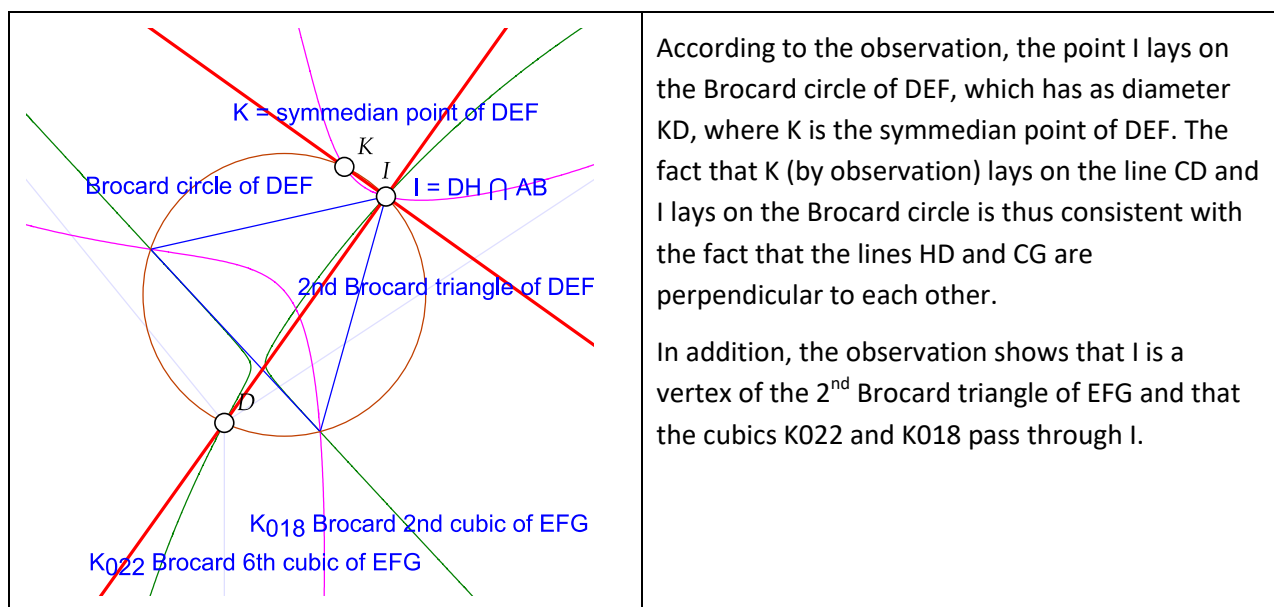


Figure 87

## 7.5 Additional comments

When inspecting GeoGen archives, you can set several options (most of which apply when a new archive is read):

**Description of constructions:** Can be set to *Symbolic* (see Figure 85 left) or *Narrative* (see Figure 85 right).

**Bleach auxiliary objects:** When this option is enabled, you can control the visibility of auxiliary objects with Alt+MouseScroll. Some figures can have several overlapping auxiliary objects, this can cause nuisance for some operations if such a construction is imported into OK Geometry. The X button in the upper right corner of the OK Geometry window can be used to ignore auxiliary objects.

**Emphasise theorem objects:** When this option is enabled, you can control the emphasis of theorem objects with Shift+MouseScroll.

**Include source data of constructions:** If this option is set, the descriptions of the configurations will contain the information from which GeoGen archive and which example in the archive is the construction from.

**Show also simple problems:** When this option is enabled, GeoGen generates a large archive with prefix 'output\_' and extension '.txt'. The archive contains besides non-trivial problems (with relevance > 10 000) also easier tasks (with relevance  $\geq 9200$ ). The relevance of simple problems is related to the number of inferences GeoGen used in the proof.

**Relevance limit:** Only tasks-configurations for which the relevance exceed this limit will be displayed.

There is a certain similarity in the creation of GeoGen archives of non-trivial proof tasks and generic constructions used in OK Geometry. But the similarity is rather superficial. Generic constructions are families of constructionally isomorphic configurations based on a large number of admissible operations. In such families one can observe and search for different properties. Unless the searched properties are bound to specific instances of objects, the observations are slow. On the other hand, GeoGen can deal with larger families of configurations that are freely generated with a limited number of specific operations. GeoGen observes fewer properties, but takes into account all objects. GeoGen is not only is very fast in observing, but determines which observed facts of configurations are non-trivial by proving or attempting to prove the observed facts.